Comparative Smootheology

Workshop on Loops, Strings and Moduli Spaces, Chern Institute of Math, Tianjin, China

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Contents

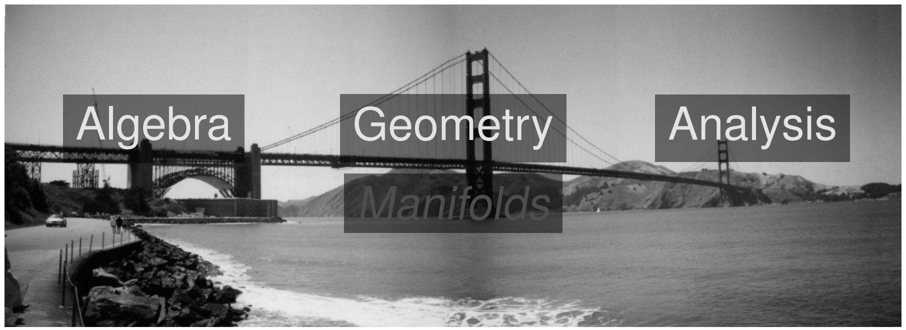
Smootheology

There now follows a party political broadcast for the

Frölicher Party

Manifolds

Manifolds are great, it's a pity more things aren't manifolds.



Goal: build a wider bridge.

Extending the Foundations

Question
What is a manifold?

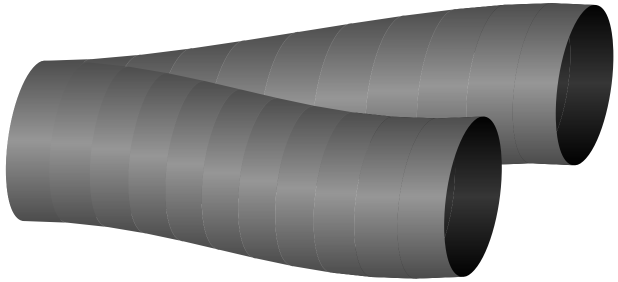
Answer
Locally diffeomorphic to a model space.

Question
How do we extend "manifolds"?

Answer
Find more models.

Manifolds with Boundary

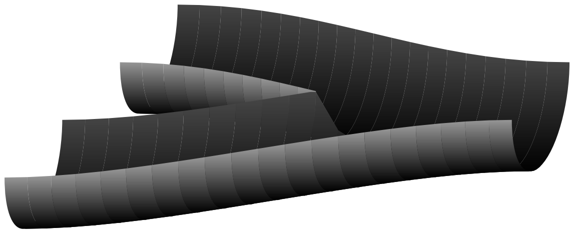
Want to study cobordisms.



New model: half spaces.

Manifolds with Corners

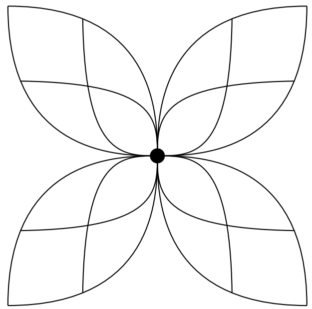
Want to study cobordisms of cobordisms.



New model: quadrants.

Manifolds with Singularities

Want to study smash products.



New model: singular points.

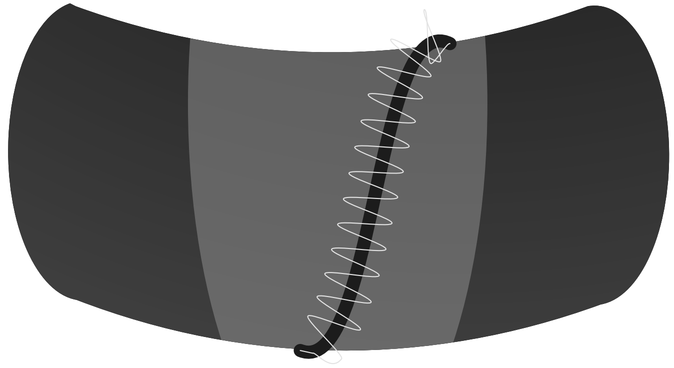
Interlude

The models get more complicated faster than the spaces.

> *The only thing capable of modelling the universe is the universe itself.*
>
> *D. Hume*

Infinite Dimensional Manifolds

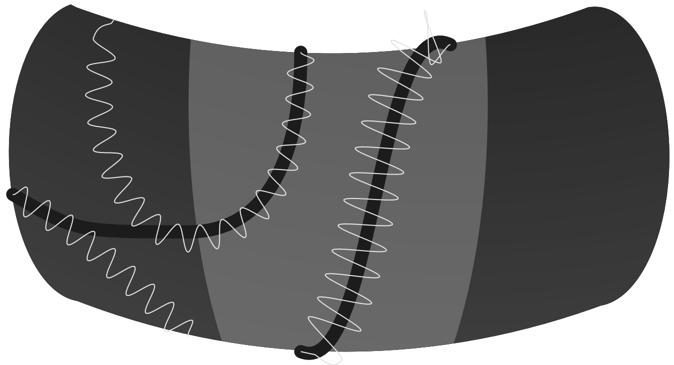
Want to study loop spaces.



New model: loops in $\mathbb{R}^n$.

Not Manifolds at all

Want to study path spaces.



No good local models.

Back to the Drawing Board

Question
How do we extend “manifolds”?

Answer
What answer do we want?

Should have:

- Subobjects
- Quotients
- Mapping spaces

All **Categorical** in nature.

The Search

Looking for a:

C^5:
complete,
co-complete,
cartesian closed
category.

Guiding principle:

The Smooth,
the Whole Smooth,
and Nothing But the Smooth.

The Smooth: Building a Category

Question
What is a manifold?

Categorical Answer
The structure needed to define smooth maps.

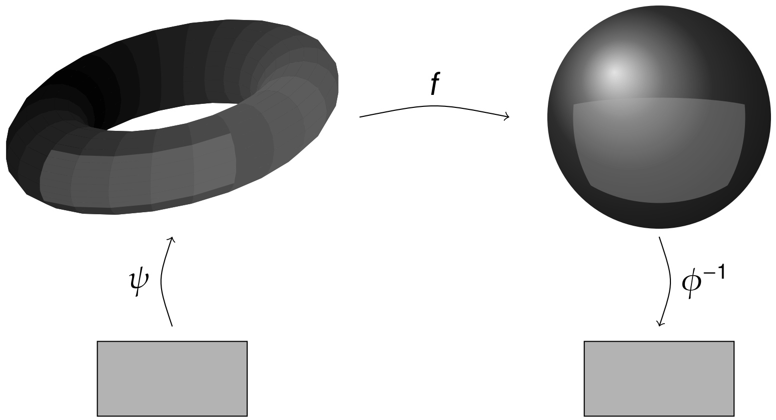
A category has

Objects

and

Morphisms

Morphism of Manifolds



$f\colon M \to N$ smooth if $\phi^{-1} f \psi$ is C^∞

The Role of the Charts

- Charts **control** smooth structure
- Charts provide **tests** or **probes**
- Map is **smooth** if it looks smooth when we **test** it

The role of **charts** is to transport the question of smoothness to **familiar** spaces.
Replace **local models** by **test spaces**.
Retain:

$$\mathcal{I}(U) = \{\psi\colon \mathbb{R}^m \supseteq U \to M\} \quad \text{input test functions}$$
$$\mathcal{O}(V; \mathbb{R}^m) = \{\phi\colon M \supseteq V \to \mathbb{R}^m\} \quad \text{output test functions}$$

Note: not homeomorphisms

First Candidate

Definition (First Attempt)

A **smooth space** is a triple $(X, \mathcal{I}, \mathcal{O})$ where:

- X is a **topological space**
- $\mathcal{I}(U) \subseteq \text{Top}(U, X)$, $U \subseteq \mathbb{R}^m$ open,
- $\mathcal{O}(V; \mathbb{R}^m) \subseteq \text{Top}(V, \mathbb{R}^m)$, $V \subseteq X$ open.

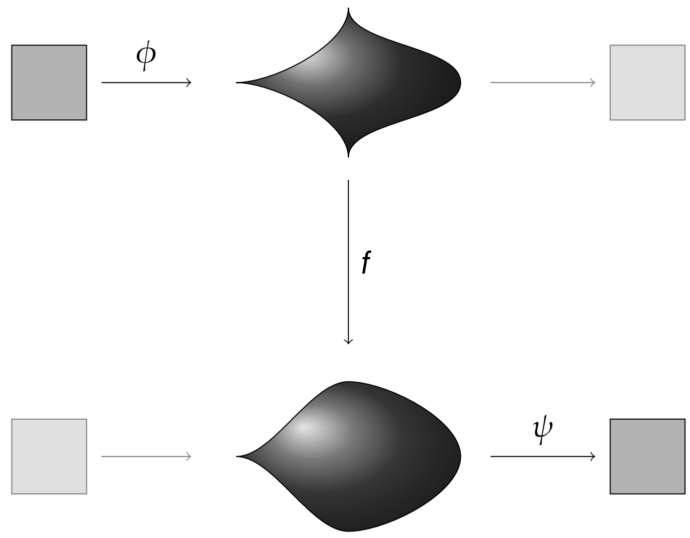
A **morphism** is a continuous map $f\colon X \to Y$ such that

$$\phi f \psi \text{ is } C^\infty \text{ for } \psi \in \mathcal{I}(U), \phi \in \mathcal{O}(V; \mathbb{R}^m)$$

Notation:

- smooth map = morphism
- $\psi \in \mathcal{I}$, $\phi \in \mathcal{O}$, $\theta \in C^\infty$

Morphisms



Categorical Construct?

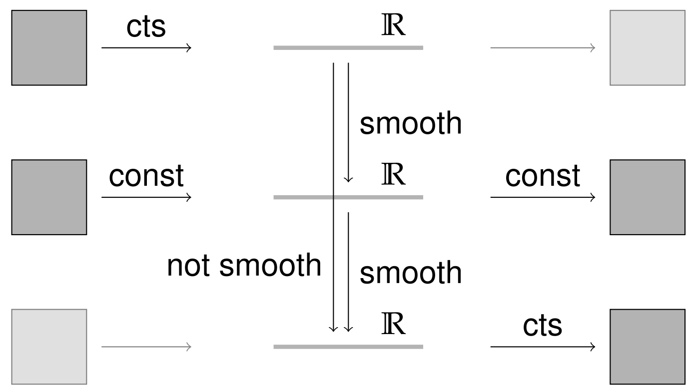
1. Identities:
   $1_X\colon X \to X$ must be smooth.

$$\psi \in \mathcal{I}, \phi \in \mathcal{O} \text{ then } \phi\psi = \phi 1_X \psi \in C^\infty$$

Compatibility Condition

Categorical Construct?

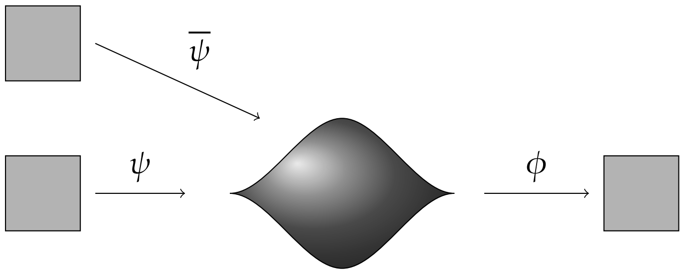
2. Composition:



Composition

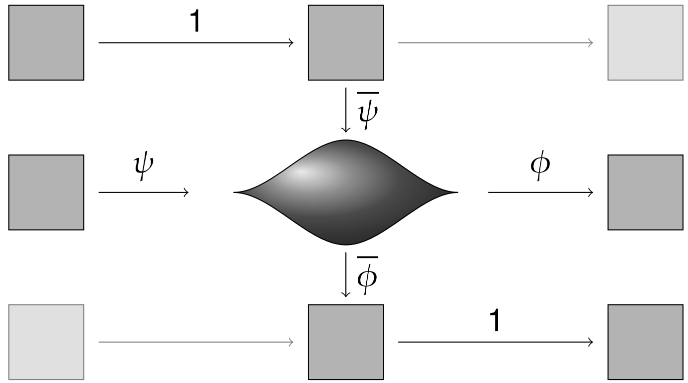
Definition
The **completed** inputs and outputs are

1. $\overline{\mathcal{I}}(U) = \{\overline{\psi} \colon U \to X : \phi\overline{\psi} \in C^\infty, \phi \in \mathcal{O}\}$
2. $\overline{\mathcal{O}}(V; \mathbb{R}^m) = \{\overline{\phi} \colon V \to \mathbb{R}^m : \overline{\phi}\psi \in C^\infty, \psi \in \mathcal{I}\}$



Composition

Proposition
*Composition is well-defined for smooth objects where $(\overline{\mathcal{I}}, \overline{\mathcal{O}})$ satisfies **compatibility**.*



Second Candidate

Definition (Second Attempt)
A **smooth space** is a triple $(X, \mathcal{I}, \mathcal{O})$ where:

- X is a **topological space**
- $\mathcal{I}(U) \subseteq \mathrm{Top}(U, X)$, $U \subseteq \mathbb{R}^m$ open,
- $\mathcal{O}(V; \mathbb{R}^m) \subseteq \mathrm{Top}(V, \mathbb{R}^m)$, $V \subseteq X$ open.

such that

- $\mathcal{I}$ and $\mathcal{O}$ are **compatible**,
- $\overline{\mathcal{I}}$ and $\overline{\mathcal{O}}$ are **also compatible**.

A **morphism** is a continuous map $f \colon X \to Y$ such that

$$\phi f \psi \in C^\infty \text{ for } \psi \in \mathcal{I}, \phi \in \mathcal{O}$$

The Whole Smooth: Too Many Smooth Spaces Spoil The Category

Lemma
*$(X, \mathcal{I}, \mathcal{O})$ smooth space then $(X, \overline{\mathcal{I}}, \overline{\mathcal{O}})$ also smooth space **and moreover***

$$1_X \colon (X, \mathcal{I}, \mathcal{O}) \to (X, \overline{\mathcal{I}}, \overline{\mathcal{O}})$$

is an isomorphism.

Proof.

1. $\overline{\overline{\mathcal{I}}} = \overline{\mathcal{I}}$ and $\overline{\overline{\mathcal{O}}} = \overline{\mathcal{O}}$
2. $\phi\psi \in C^\infty$ for $\phi \in \overline{\mathcal{O}}$, $\psi \in \mathcal{I}$ and for $\phi \in \mathcal{O}$, $\psi \in \overline{\mathcal{I}}$ □

Third Candidate

Definition (Third Attempt)
A **smooth space** is a triple $(X, \mathcal{I}, \mathcal{O})$ where:

- X is a **topological space**
- $\mathcal{I}(U) \subseteq \text{Top}(U, X)$, $U \subseteq \mathbb{R}^m$ open,
- $\mathcal{O}(V; \mathbb{R}^m) \subseteq \text{Top}(V, \mathbb{R}^m)$, $V \subseteq X$ open.

such that

- $\mathcal{I}$ and $\mathcal{O}$ are **compatible**,
- $\mathcal{I}$ and $\mathcal{O}$ are **saturated**: $\mathcal{I} = \overline{\mathcal{I}}$, $\mathcal{O} = \overline{\mathcal{O}}$.

A **morphism** is a continuous map $f\colon X \to Y$ such that

$$\phi f \psi \in C^\infty \text{ for } \psi \in \mathcal{I}, \phi \in \mathcal{O}$$

Analogy

2nd Definition versus **3rd Definition**	**smooth atlas** versus **maximal smooth atlas**

Detox

Lemma
$(X, \mathcal{I}, \mathcal{O})$ is completely determined by

- $\mathcal{O}(V; \mathbb{R})$
- $\mathcal{I}(\mathbb{R})$

Proof.

1. $\phi\colon U \to \mathbb{R}^m$ is C^∞ if and only if each $p_i\phi\colon U \to \mathbb{R}$ is C^∞, p_i projection
2. $\psi\colon U \to \mathbb{R}^m$ is C^∞ if and only if each $\psi\gamma\colon \mathbb{R} \to \mathbb{R}^m$ is C^∞, $\gamma \in C^\infty(\mathbb{R}, U)$ [Boman's Theorem] □

Fourth Candidate

Definition (Fourth Attempt)
A **smooth space** is a triple $(X, \mathcal{I}, \mathcal{O})$ where:

- X is a **topological space**
- $\mathcal{I} \subseteq \text{Top}(\mathbb{R}, X)$,
- $\mathcal{O}(V) \subseteq \text{Top}(V, \mathbb{R})$, $V \subseteq X$ open.

such that

- $\mathcal{I}$ and $\mathcal{O}$ are **compatible**,
- $\mathcal{I}$ and $\mathcal{O}$ are **saturated**: $\mathcal{I} = \overline{\mathcal{I}}$, $\mathcal{O} = \overline{\mathcal{O}}$.

A **morphism** is a continuous map $f\colon X \to Y$ such that

$$\phi f \psi \in C^\infty \text{ for } \psi \in \mathcal{I}, \phi \in \mathcal{O}$$

Nothing But the Smooth: Chicken and Egg

Question
Is “Smooth” built **on top** of “Continuous” or **alongside**?

- Topology **only** used for **local** functions.
- Can get topology from $\mathcal{I}$ and $\mathcal{O}$.
- Topology from $\mathcal{O}$ has **bump functions**.
- **Local** functions extend **globally**.

Fifth Candidate

Definition (Fifth Attempt)
A **smooth space** is a triple $(X, \mathcal{I}, \mathcal{O})$ where:

- X is a **set**
- $\mathcal{I} \subseteq \mathrm{Set}(\mathbb{R}, X)$,
- $\mathcal{O} \subseteq \mathrm{Set}(X, \mathbb{R})$.

such that

- $\mathcal{I}$ and $\mathcal{O}$ are **compatible**,
- $\mathcal{I}$ and $\mathcal{O}$ are **saturated**: $\mathcal{I} = \overline{\mathcal{I}}$, $\mathcal{O} = \overline{\mathcal{O}}$.

A **morphism** is a map $f \colon X \to Y$ such that

$$\phi f \psi \in C^\infty \text{ for } \psi \in \mathcal{I}, \phi \in \mathcal{O}$$

Fifth Candidate: Frölicher Space

Definition (Frölicher Space)
A **Frölicher space** is a triple $(X, \mathcal{C}, \mathcal{F})$ where:

- X is a **set**
- $\mathcal{C} \subseteq \mathrm{Set}(\mathbb{R}, X)$,
- $\mathcal{F} \subseteq \mathrm{Set}(X, \mathbb{R})$.

such that

- $\mathcal{C} = \{\psi \colon \mathbb{R} \to X : \phi\psi \in C^\infty(\mathbb{R}, \mathbb{R}), \phi \in \mathcal{F}\}$
- $\mathcal{F} = \{\phi \colon X \to \mathbb{R} : \phi\psi \in C^\infty(\mathbb{R}, \mathbb{R}), \psi \in \mathcal{C}\}$

A **morphism** is a map $f \colon X \to Y$ such that

$$\phi f \psi \in C^\infty \text{ for } \psi \in \mathcal{C}, \phi \in \mathcal{F}$$

Smootheology

Theorem (Frölicher)
The category of Frölicher spaces is a

complete, co-complete, cartesian closed category.

Which was what we wanted!

Conclusion (Part I)
By focussing on **morphisms** and looking for **simplicity** we found a natural path to **Frölicher spaces**.

Comparative

- Frölicher spaces are not the only candidate.
- Other versions:

K. T. Chen	4 versions!
J. M. Souriau	diffeological spaces
J. W. Smith	
R. Sikorski	
D. Spivak	derived smooth manifolds
J. Lurie	structured space
A. Kriegl and P. Michor	
M. Kreck	stratifold
H. Hofer	polyfold
	orbifold
	differentiable stack

More?

Comparitive

Use

- local models
- test spaces
- both

In *Comparative Smootheology* looked at:

Test spaces + set-based

Frölicher, Chen, Souriau, Smith, Sikorski.

Schematic

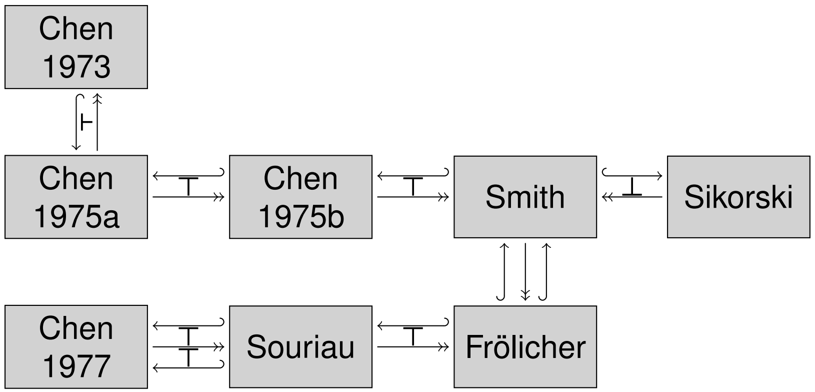


Figure: The relationships between the categories

Characteristics of Frölicher Spaces

Cons

- Not locally cartesian closed
- Not easy to make non-set based

Characteristics of Frölicher Spaces

Pros

- Inclusion of manifolds is limit and colimit preserving. (into *Hausdoff* Frölicher spaces)
- "Smallest" extension

What Next?

- What of differential topology/geometry extends to Frölicher spaces?
- What extra structure is needed for those bits that don't extend?

Example

Tangent "spaces" extend but are not vector spaces.
Need extra structure to get addition.

Conclusion

- **Frölicher spaces** arise by taking seriously the notion of a **morphism of smooth manifolds**.
- The various categories fit into a neat setting but are **distinct**. So may have **distinct** behaviour.
- Examining what does and does not extend sheds light on the role of **smoothness** in differential topology.

Further Information

My homepage	Andrew Stacey NTNU	Search
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