Comparative Smootheology

Workshop on Loops, Strings and Moduli Spaces, Chern Institute of Math, Tianjin, China

Andrew Stacey

Norges Teknisk-Naturvitenskapelige Universitet
Trondheim

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Contents

1. Smootheology

   Aims:

   1.1 Motivate generalising manifolds.
   1.2 Frölicher spaces are “obvious” generalisation of manifolds.

2. Comparative

   Aim:

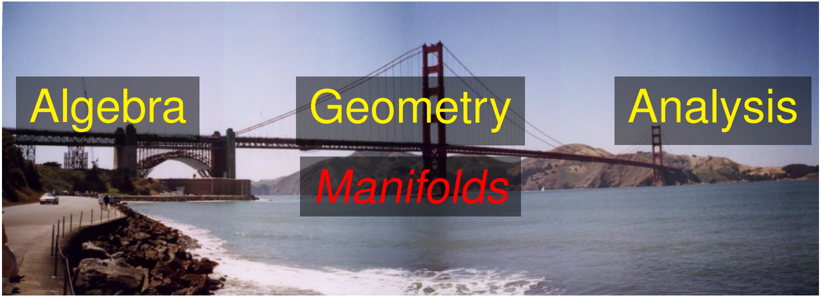
   2.1 Show how the various categories of “smooth objects” fit together.
   2.2 What is special about Frölicher spaces.

There now follows a party political broadcast
for the

Frölicher Party

Manifolds

Manifolds are great, it's a pity more things aren't manifolds.



Goal: build a wider bridge.

Extending the Foundations

Question
What is a manifold?

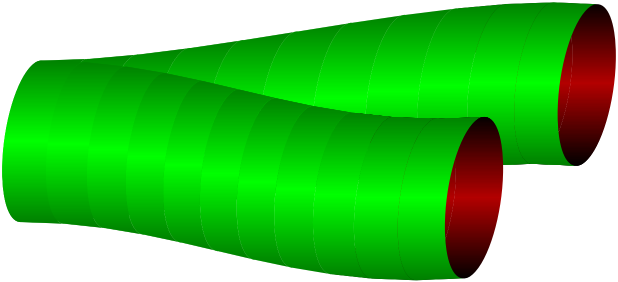
Answer
Locally diffeomorphic to a model space.

Question
How do we extend "manifolds"?

Answer
Find more models.

Manifolds with Boundary

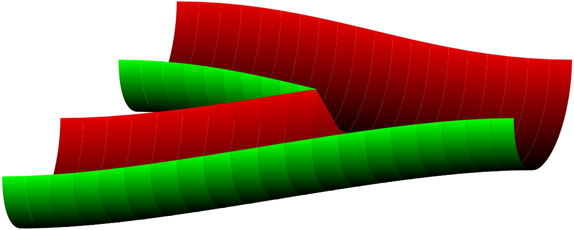
Want to study cobordisms.



New model: half spaces.

Manifolds with Corners

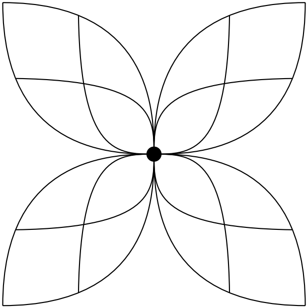
Want to study cobordisms of cobordisms.



New model: quadrants.

Manifolds with Singularities

Want to study smash products.



New model: singular points.

Interlude

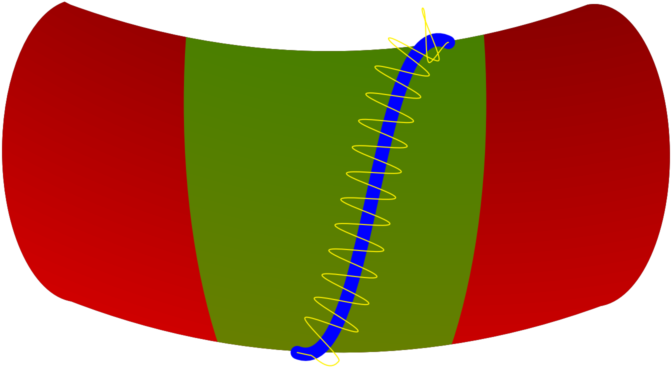
The models get more complicated faster than the spaces.

The only thing capable of modelling the universe is the universe itself.

D. Hume

Infinite Dimensional Manifolds

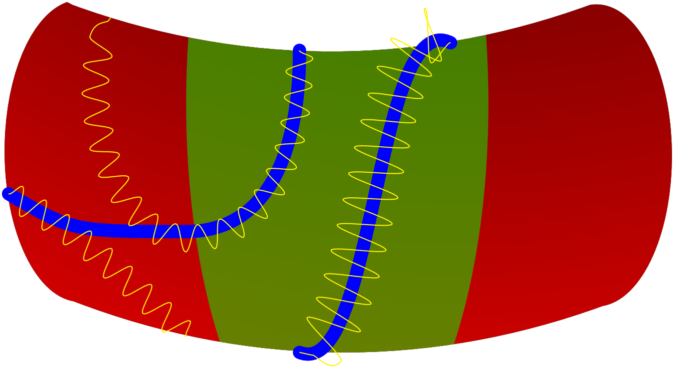
Want to study loop spaces.



New model: loops in $\mathbb{R}^n$.

Not Manifolds at all

Want to study path spaces.



No good local models.

Back to the Drawing Board

Question

How do we extend “manifolds”?

Answer

What answer do we want?

Should have:

- Subobjects
- Quotients
- Mapping spaces

All Categorical in nature.

The Search

Looking for a:

C^5:
complete,
co-complete,
cartesian closed
category.

Guiding principle:

The Smooth,
the Whole Smooth,
and Nothing But the Smooth.

The Smooth

Building a Category

Question

What is a manifold?

Categorical Answer

The structure needed to define smooth maps.

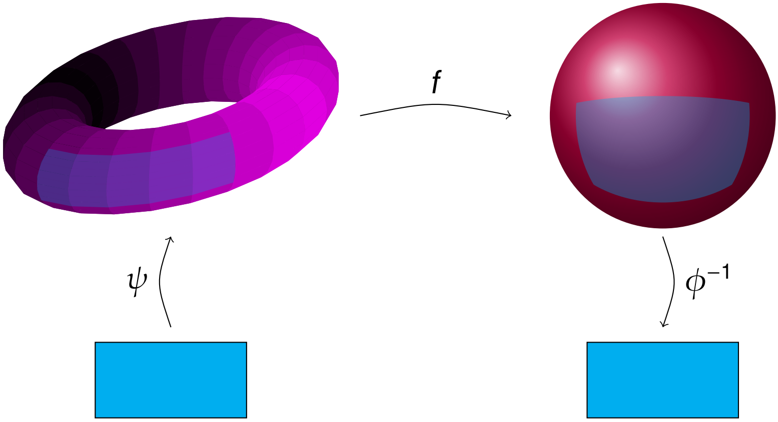
A category has

Objects

and

Morphisms

Morphism of Manifolds



$f\colon M \to N$ smooth if $\phi^{-1} f \psi$ is C^∞

The Role of the Charts

- Charts control smooth structure
- Charts provide tests or probes
- Map is smooth if it looks smooth when we test it

The role of charts is to transport the question of smoothness to familiar spaces.
Replace local models by test spaces.
Retain:

$$\mathcal{I}(U) = \{\psi \colon \mathbb{R}^m \supseteq U \to M\} \quad \text{input test functions}$$

$$\mathcal{O}(V; \mathbb{R}^m) = \{\phi \colon M \supseteq V \to \mathbb{R}^m\} \quad \text{output test functions}$$

Note: not homeomorphisms

First Candidate

Definition (First Attempt)

A smooth space is a triple $(X, \mathcal{I}, \mathcal{O})$ where:

- X is a topological space
- $\mathcal{I}(U) \subseteq \operatorname{Top}(U, X)$, $U \subseteq \mathbb{R}^m$ open,
- $\mathcal{O}(V; \mathbb{R}^m) \subseteq \operatorname{Top}(V, \mathbb{R}^m)$, $V \subseteq X$ open.

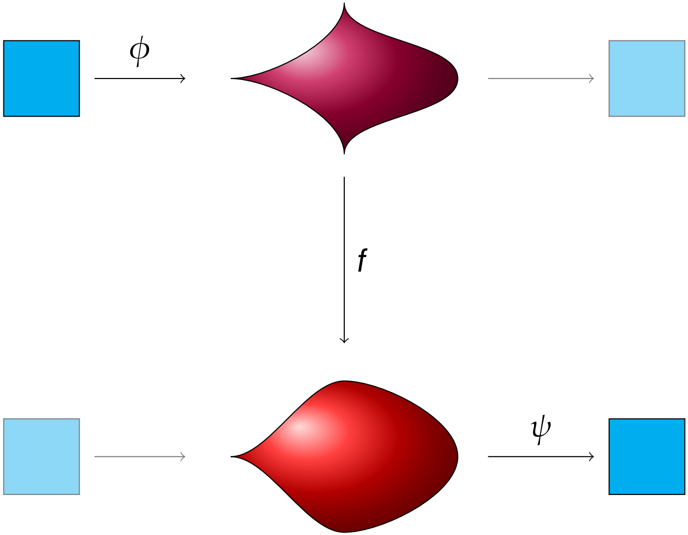
A morphism is a continuous map $f \colon X \to Y$ such that

$$\phi f \psi \text{ is } C^\infty \text{ for } \psi \in \mathcal{I}(U), \phi \in \mathcal{O}(V; \mathbb{R}^m)$$

Notation:

- smooth map = morphism
- $\psi \in \mathcal{I}$, $\phi \in \mathcal{O}$, $\theta \in C^\infty$

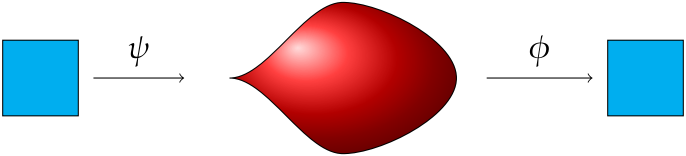
Morphisms



Categorical Construct?

1. Identities:
   $1_X \colon X \to X$ must be smooth.

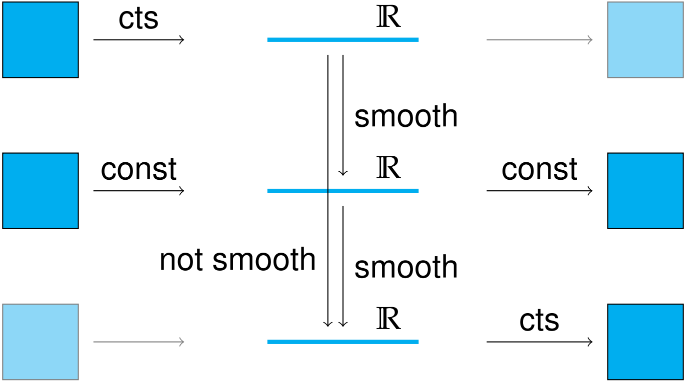
$$\psi \in \mathcal{I}, \phi \in \mathcal{O} \text{ then } \phi\psi = \phi 1_X \psi \in C^\infty$$



Compatibility Condition

Categorical Construct?

2. Composition:

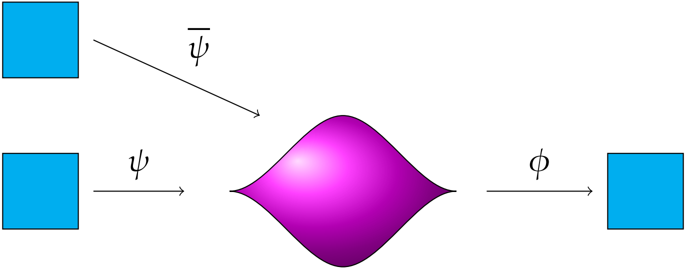


Composition

Definition

The completed inputs and outputs are

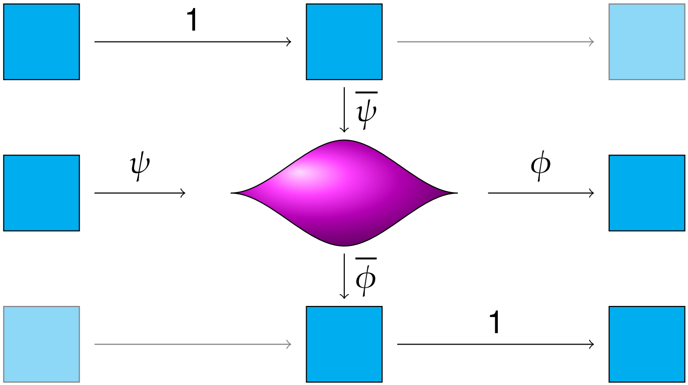
1. $\overline{\mathcal{I}}(U) = \{\overline{\psi} \colon U \to X : \phi\overline{\psi} \in C^\infty, \phi \in \mathcal{O}\}$
2. $\overline{\mathcal{O}}(V; \mathbb{R}^m) = \{\overline{\phi} \colon V \to \mathbb{R}^m : \overline{\phi}\psi \in C^\infty, \psi \in \mathcal{I}\}$



Composition

Proposition

Composition is well-defined for smooth objects where $(\overline{\mathcal{I}}, \overline{\mathcal{O}})$ satisfies compatibility.



Second Candidate

Definition (Second Attempt)

A smooth space is a triple $(X, \mathcal{I}, \mathcal{O})$ where:

- X is a topological space
- $\mathcal{I}(U) \subseteq \mathrm{Top}(U, X)$, $U \subseteq \mathbb{R}^m$ open,
- $\mathcal{O}(V; \mathbb{R}^m) \subseteq \mathrm{Top}(V, \mathbb{R}^m)$, $V \subseteq X$ open.

such that

- $\mathcal{I}$ and $\mathcal{O}$ are compatible,
- $\overline{\mathcal{I}}$ and $\overline{\mathcal{O}}$ are also compatible.

A morphism is a continuous map $f \colon X \to Y$ such that

$$\phi f \psi \in C^\infty \text{ for } \psi \in \mathcal{I}, \phi \in \mathcal{O}$$

The Whole Smooth

Too Many Smooth Spaces Spoil The Category

Lemma

$(X, \mathcal{I}, \mathcal{O})$ smooth space then $(X, \overline{\mathcal{I}}, \overline{\mathcal{O}})$ also smooth space *and moreover*

$$1_X \colon (X, \mathcal{I}, \mathcal{O}) \to (X, \overline{\mathcal{I}}, \overline{\mathcal{O}})$$

is an isomorphism.

Proof.

1. $\overline{\overline{\mathcal{I}}} = \overline{\mathcal{I}}$ and $\overline{\overline{\mathcal{O}}} = \overline{\mathcal{O}}$
2. $\phi\psi \in C^\infty$ for $\phi \in \overline{\mathcal{O}}$, $\psi \in \mathcal{I}$ and for $\phi \in \mathcal{O}$, $\psi \in \overline{\mathcal{I}}$ □

Third Candidate

Definition (Third Attempt)

A smooth space is a triple $(X, \mathcal{I}, \mathcal{O})$ where:

- X is a topological space
- $\mathcal{I}(U) \subseteq \operatorname{Top}(U, X)$, $U \subseteq \mathbb{R}^m$ open,
- $\mathcal{O}(V; \mathbb{R}^m) \subseteq \operatorname{Top}(V, \mathbb{R}^m)$, $V \subseteq X$ open.

such that

- $\mathcal{I}$ and $\mathcal{O}$ are compatible,
- $\mathcal{I}$ and $\mathcal{O}$ are saturated: $\mathcal{I} = \overline{\mathcal{I}}$, $\mathcal{O} = \overline{\mathcal{O}}$.

A morphism is a continuous map $f \colon X \to Y$ such that

$$\phi f \psi \in C^\infty \text{ for } \psi \in \mathcal{I}, \phi \in \mathcal{O}$$

Analogy

2nd Definition	smooth atlas
versus	versus
3rd Definition	maximal smooth atlas

Detox

Lemma

$(X, \mathcal{I}, \mathcal{O})$ is completely determined by

- $\mathcal{O}(V; \mathbb{R})$
- $\mathcal{I}(\mathbb{R})$

Proof.

1. $\phi \colon U \to \mathbb{R}^m$ is C^∞ if and only if each $p_i \phi \colon U \to \mathbb{R}$ is C^∞, p_i projection
2. $\psi \colon U \to \mathbb{R}^m$ is C^∞ if and only if each $\psi\gamma \colon \mathbb{R} \to \mathbb{R}^m$ is C^∞, $\gamma \in C^\infty(\mathbb{R}, U)$ [Boman's Theorem] □

Fourth Candidate

Definition (Fourth Attempt)

A smooth space is a triple $(X, \mathcal{I}, \mathcal{O})$ where:

- X is a topological space
- $\mathcal{I} \subseteq \mathrm{Top}(\mathbb{R}, X)$,
- $\mathcal{O}(V) \subseteq \mathrm{Top}(V, \mathbb{R})$, $V \subseteq X$ open.

such that

- $\mathcal{I}$ and $\mathcal{O}$ are compatible,
- $\mathcal{I}$ and $\mathcal{O}$ are saturated: $\mathcal{I} = \overline{\mathcal{I}}$, $\mathcal{O} = \overline{\mathcal{O}}$.

A morphism is a continuous map $f \colon X \to Y$ such that

$$\phi f \psi \in C^\infty \text{ for } \psi \in \mathcal{I}, \phi \in \mathcal{O}$$

Nothing But The Smooth

Chicken and Egg

Question

Is “Smooth” built on top of “Continuous” or alongside?

- Topology only used for local functions.
- Can get topology from $\mathcal{I}$ and $\mathcal{O}$.
- Topology from $\mathcal{O}$ has bump functions.
- Local functions extend globally.

Fifth Candidate

Definition (Fifth Attempt)

A smooth space is a triple $(X, \mathcal{I}, \mathcal{O})$ where:

- X is a set
- $\mathcal{I} \subseteq \operatorname{Set}(\mathbb{R}, X)$,
- $\mathcal{O} \subseteq \operatorname{Set}(X, \mathbb{R})$.

such that

- $\mathcal{I}$ and $\mathcal{O}$ are compatible,
- $\mathcal{I}$ and $\mathcal{O}$ are saturated: $\mathcal{I} = \overline{\mathcal{I}}$, $\mathcal{O} = \overline{\mathcal{O}}$.

A morphism is a map $f \colon X \to Y$ such that

$$\phi f \psi \in C^\infty \text{ for } \psi \in \mathcal{I}, \phi \in \mathcal{O}$$

Fifth Candidate: Frölicher Space

Definition (Frölicher Space)

A Frölicher space is a triple $(X, \mathcal{C}, \mathcal{F})$ where:

- X is a set
- $\mathcal{C} \subseteq \mathrm{Set}(\mathbb{R}, X)$,
- $\mathcal{F} \subseteq \mathrm{Set}(X, \mathbb{R})$.

such that

- $\mathcal{C} = \{\psi \colon \mathbb{R} \to X : \phi\psi \in C^\infty(\mathbb{R}, \mathbb{R}), \phi \in \mathcal{F}\}$
- $\mathcal{F} = \{\phi \colon X \to \mathbb{R} : \phi\psi \in C^\infty(\mathbb{R}, \mathbb{R}), \psi \in \mathcal{C}\}$

A morphism is a map $f \colon X \to Y$ such that

$$\phi f \psi \in C^\infty \text{ for } \psi \in \mathcal{C}, \phi \in \mathcal{F}$$

Smootheology

Theorem (Frölicher)

The category of Frölicher spaces is a

complete, co-complete, cartesian closed category.

Which was what we wanted!

Conclusion (Part I)

By focussing on morphisms and looking for simplicity we found a natural path to Frölicher spaces.

Comparative

- Frölicher spaces are not the only candidate.
- Other versions:

K. T. Chen	4 versions!
J. M. Souriau	diffeological spaces
J. W. Smith	
R. Sikorski	
D. Spivak	derived smooth manifolds
J. Lurie	structured space
A. Kriegl and P. Michor	
M. Kreck	stratifold
H. Hofer	polyfold
	orbifold
	differentiable stack

More?

Comparitive

Use

- local models
- test spaces
- both

In *Comparative Smootheology* looked at:

Test spaces + set-based

Frölicher, Chen, Souriau, Smith, Sikorski.

Schematic

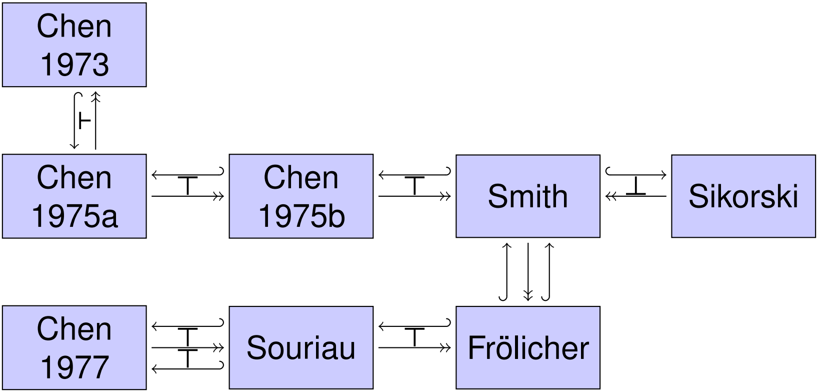


Figure: The relationships between the categories

Characteristics of Frölicher Spaces

Cons

- Not locally cartesian closed
- Not easy to make non-set based

Characteristics of Frölicher Spaces

Pros

- Inclusion of manifolds is limit and colimit preserving. (into *Hausdoff* Frölicher spaces)
- “Smallest” extension

What Next?

- What of differential topology/geometry extends to Frölicher spaces?
- What extra structure is needed for those bits that don't extend?

Example

Tangent "spaces" extend but are not vector spaces.
Need extra structure to get addition.

Conclusion

- Frölicher spaces arise by taking seriously the notion of a morphism of smooth manifolds.
- The various categories fit into a neat setting but are distinct. So may have distinct behaviour.
- Examining what does and does not extend sheds light on the role of smoothness in differential topology.

Further Information

My homepage	Andrew Stacey NTNU	Search
n—lab	nlab	Search