Comparative Smootheology
Workshop on Loops, Strings and Moduli Spaces,
Chern Institute of Math, Tianjin, China

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1. Smootheology
   Aims:
   1.1 Motivate generalising manifolds.
   1.2 Frölicher spaces are “obvious” generalisation of manifolds.

2. Comparative
   Aim:
   2.1 Show how the various categories of “smooth objects” fit together.
   2.2 What is special about Frölicher spaces.
There now follows a party political broadcast for the Frölicher Party.
Manifolds

*Manifolds are great, it’s a pity more things aren’t manifolds.*

**Goal:** build a wider bridge.
Question
What is a manifold?

Answer
Locally diffeomorphic to a model space.

Question
How do we extend “manifolds”?

Answer
Find more models.
Manifolds with Boundary

Want to study cobordisms.

New model: half spaces.
Manifolds with Corners

Want to study cobordisms of cobordisms.

New model: quadrants.
Manifolds with Singularities

Want to study smash products.

New model: singular points.
The models get more complicated faster than the spaces.

_The only thing capable of modelling the universe is the universe itself._

*D. Hume*
Infinite Dimensional Manifolds

Want to study loop spaces.

New model: loops in $\mathbb{R}^n$. 

Not Manifolds at all

Want to study path spaces.

No good local models.
Question
How do we extend “manifolds”?

Answer
What answer do we want?
Should have:
  ▶ Subobjects
  ▶ Quotients
  ▶ Mapping spaces
All Categorical in nature.
Looking for a:

\[ C^5: \]
complete,  
co-complete,  
cartesian closed  
category.

Guiding principle:

The Smooth,  
the Whole Smooth,  
and Nothing But the Smooth.
The Smooth
Building a Category

**Question**
What is a manifold?

**Categorical Answer**
The structure needed to define smooth maps.
A category has

Objects
and

Morphisms
Morphism of Manifolds

$f : M \rightarrow N$ smooth if $\phi^{-1} f \psi$ is $C^\infty$
The Role of the Charts

- Charts control smooth structure
- Charts provide tests or probes
- Map is smooth if it looks smooth when we test it

The role of charts is to transport the question of smoothness to familiar spaces.
Replace local models by test spaces.
Retain:

\[ I(U) = \{ \psi : \mathbb{R}^m \supseteq U \to M \} \quad \text{input test functions} \]
\[ O(V; \mathbb{R}^m) = \{ \phi : M \supseteq V \to \mathbb{R}^m \} \quad \text{output test functions} \]

Note: not homeomorphisms
Definition (First Attempt)

A smooth space is a triple $(X, \mathcal{I}, \mathcal{O})$ where:

- $X$ is a topological space
- $\mathcal{I}(U) \subseteq \text{Top}(U, X)$, $U \subseteq \mathbb{R}^m$ open,
- $\mathcal{O}(V; \mathbb{R}^m) \subseteq \text{Top}(V, \mathbb{R}^m)$, $V \subseteq X$ open.

A morphism is a continuous map $f : X \rightarrow Y$ such that

$$\phi f \psi \text{ is } C^\infty \text{ for } \psi \in \mathcal{I}(U), \phi \in \mathcal{O}(V; \mathbb{R}^m)$$

Notation:

- smooth map = morphism
- $\psi \in \mathcal{I}$, $\phi \in \mathcal{O}$, $\theta \in C^\infty$
Morphisms

\[ \phi \quad \rightarrow \quad f \quad \rightarrow \quad \psi \]
Categorical Construct?

1. Identities:
   \( 1_X : X \rightarrow X \) must be smooth.

\[ \psi \in \mathcal{I}, \phi \in \mathcal{O} \text{ then } \phi \psi = \phi 1_X \psi \in \mathcal{C}^\infty \]
Categorical Construct?

2. Composition:

\[ \text{cts} \rightarrow \mathbb{R} \]

\[ \text{const} \rightarrow \mathbb{R} \]

not smooth

\[ \text{smooth} \rightarrow \mathbb{R} \]

\[ \text{smooth} \rightarrow \mathbb{R} \]

\[ \text{cts} \rightarrow \mathbb{R} \]

\[ \text{const} \rightarrow \mathbb{R} \]
Composition

**Definition**

The completed inputs and outputs are

1. $\overline{I}(U) = \{\overline{\psi} : U \to X : \phi \overline{\psi} \in C^\infty, \phi \in \mathcal{O}\}$
2. $\overline{O}(V; \mathbb{R}^m) = \{\overline{\phi} : V \to \mathbb{R}^m : \overline{\phi} \psi \in C^\infty, \psi \in \mathcal{I}\}$

![Diagram](attachment://composition_diagram.png)
Composition

Proposition

Composition is well-defined for smooth objects where \((\overline{I}, \overline{O})\) satisfies compatibility.
Definition (Second Attempt)

A smooth space is a triple $(X, \mathcal{I}, \mathcal{O})$ where:

- $X$ is a topological space
- $\mathcal{I}(U) \subseteq \text{Top}(U, X), \ U \subseteq \mathbb{R}^m$ open,
- $\mathcal{O}(V; \mathbb{R}^m) \subseteq \text{Top}(V, \mathbb{R}^m), \ V \subseteq X$ open.

such that

- $\mathcal{I}$ and $\mathcal{O}$ are compatible,
- $\overline{\mathcal{I}}$ and $\overline{\mathcal{O}}$ are also compatible.

A morphism is a continuous map $f: X \to Y$ such that

$$\phi f \psi \in C^\infty \text{ for } \psi \in \mathcal{I}, \phi \in \mathcal{O}$$
The Whole Smooth
Lemma

\((X, \mathcal{I}, \mathcal{O})\) smooth space then \((X, \overline{\mathcal{I}}, \overline{\mathcal{O}})\) also smooth space and moreover

\[1_X: (X, \mathcal{I}, \mathcal{O}) \rightarrow (X, \overline{\mathcal{I}}, \overline{\mathcal{O}})\]

is an isomorphism.

Proof.

1. \(\overline{\overline{\mathcal{I}}} = \overline{\mathcal{I}}\) and \(\overline{\overline{\mathcal{O}}} = \overline{\mathcal{O}}\)

2. \(\phi \psi \in C^\infty\) for \(\phi \in \overline{\mathcal{O}}, \psi \in \mathcal{I}\) and for \(\phi \in \mathcal{O}, \psi \in \overline{\mathcal{I}}\) \(\square\)
Definition (Third Attempt)

A smooth space is a triple \((X, \mathcal{I}, \mathcal{O})\) where:

- \(X\) is a topological space
- \(\mathcal{I}(U) \subseteq \text{Top}(U, X), U \subseteq \mathbb{R}^m\) open,
- \(\mathcal{O}(V; \mathbb{R}^m) \subseteq \text{Top}(V, \mathbb{R}^m), V \subseteq X\) open.

such that

- \(\mathcal{I}\) and \(\mathcal{O}\) are compatible,
- \(\mathcal{I}\) and \(\mathcal{O}\) are saturated: \(\mathcal{I} = \overline{\mathcal{I}}, \mathcal{O} = \overline{\mathcal{O}}\).

A morphism is a continuous map \(f: X \to Y\) such that

\[ \phi f \psi \in C^\infty \text{ for } \psi \in \mathcal{I}, \phi \in \mathcal{O} \]
Analogy

2nd Definition versus 3rd Definition

smooth atlas versus maximal smooth atlas
Lemma

$(X, I, O)$ is completely determined by

- $O(V; \mathbb{R})$
- $I(\mathbb{R})$

Proof.

1. $\phi: U \to \mathbb{R}^m$ is $C^\infty$ if and only if each $p_i\phi: U \to \mathbb{R}$ is $C^\infty$, $p_i$ projection

2. $\psi: U \to \mathbb{R}^m$ is $C^\infty$ if and only if each $\psi\gamma: \mathbb{R} \to \mathbb{R}^m$ is $C^\infty$, $\gamma \in C^\infty(\mathbb{R}, U)$ [Boman’s Theorem]
Definition (Fourth Attempt)

A smooth space is a triple \((X, \mathcal{I}, \mathcal{O})\) where:

- \(X\) is a topological space
- \(\mathcal{I} \subseteq \text{Top}(\mathbb{R}, X)\),
- \(\mathcal{O}(V) \subseteq \text{Top}(V, \mathbb{R}), \ V \subseteq X\) open.

such that

- \(\mathcal{I}\) and \(\mathcal{O}\) are compatible,
- \(\mathcal{I}\) and \(\mathcal{O}\) are saturated: \(\mathcal{I} = \overline{\mathcal{I}}, \mathcal{O} = \overline{\mathcal{O}}\).

A morphism is a continuous map \(f: X \to Y\) such that

\[ \phi f \psi \in C^\infty \text{ for } \psi \in \mathcal{I}, \phi \in \mathcal{O} \]
Nothing But The Smooth
Chicken and Egg

Question
Is “Smooth” built on top of “Continuous” or alongside?

- Topology only used for local functions.
- Can get topology from $\mathcal{I}$ and $\mathcal{O}$.
- Topology from $\mathcal{O}$ has bump functions.
- Local functions extend globally.
Fifth Candidate

Definition (Fifth Attempt)

A smooth space is a triple $(X, \mathcal{I}, \mathcal{O})$ where:

▶ $X$ is a set
▶ $\mathcal{I} \subseteq \text{Set}(\mathbb{R}, X)$,
▶ $\mathcal{O} \subseteq \text{Set}(X, \mathbb{R})$.

such that

▶ $\mathcal{I}$ and $\mathcal{O}$ are compatible,
▶ $\mathcal{I}$ and $\mathcal{O}$ are saturated: $\mathcal{I} = \overline{\mathcal{I}}, \mathcal{O} = \overline{\mathcal{O}}$.

A morphism is a map $f: X \to Y$ such that

$$\phi f \psi \in C^\infty \text{ for } \psi \in \mathcal{I}, \phi \in \mathcal{O}$$
Fifth Candidate: Frölicher Space

Definition (Frölicher Space)
A Frölicher space is a triple \((X,C,F)\) where:

- \(X\) is a set
- \(C \subseteq \text{Set}(\mathbb{R}, X)\),
- \(F \subseteq \text{Set}(X, \mathbb{R})\).

such that

- \(C = \{\psi: \mathbb{R} \rightarrow X : \phi\psi \in C^\infty(\mathbb{R}, \mathbb{R}), \phi \in F\}\)
- \(F = \{\phi: X \rightarrow \mathbb{R} : \phi\psi \in C^\infty(\mathbb{R}, \mathbb{R}), \psi \in C\}\)

A morphism is a map \(f: X \rightarrow Y\) such that

\[ \phi f\psi \in C^\infty \text{ for } \psi \in C, \phi \in F \]
Theorem (Frölicher)

The category of Frölicher spaces is a complete, co-complete, cartesian closed category.

Which was what we wanted!

Conclusion (Part I)

By focussing on morphisms and looking for simplicity we found a natural path to Frölicher spaces.
Comparative

- Frölicher spaces are not the only candidate.
- Other versions:
  - K. T. Chen
  - J. M. Souriau
diffeological spaces
  - J. W. Smith
  - R. Sikorski
  - D. Spivak
derived smooth manifolds
  - J. Lurie
  - A. Kriegl and P. Michor
  - M. Kreck
  - H. Hofer
  - orbifold
  - differentiable stack

More?
Comparative

Use

- local models
- test spaces
- both

In *Comparative Smootheology* looked at:

Test spaces + set-based

Frölicher, Chen, Souriau, Smith, Sikorski.
Figure: The relationships between the categories
Characteristics of Frölicher Spaces

Cons

- Not locally cartesian closed
- Not easy to make non-set based
Characteristics of Frölicher Spaces

Pros

- Inclusion of manifolds is limit and colimit preserving. (into Hausdoff Frölicher spaces)
- “Smallest” extension
What Next?

- What of differential topology/geometry extends to Frölicher spaces?
- What extra structure is needed for those bits that don’t extend?

Example

Tangent “spaces” extend but are not vector spaces. Need extra structure to get addition.
Conclusion

- Frölicher spaces arise by taking seriously the notion of a morphism of smooth manifolds.
- The various categories fit into a neat setting but are distinct. So may have distinct behaviour.
- Examining what does and does not extend sheds light on the role of smoothness in differential topology.

Further Information

My homepage

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