Linearity Tools

Theorem

Matrices work just the same.

Question

\[ x_1 + 2x_2 + 3x_3 = 2 \]
\[ 4x_1 + 5x_2 + 6x_3 = 5 \]

\[ x_1 + 2x_2 + 3x_3 = 2 \]
\[ 4x_1 + 5x_2 + 6x_3 = 2 \]

What is a solution of
\[ x_1 + 2x_2 + 3x_3 = 4 \]
\[ 4x_1 + 5x_2 + 6x_3 = 7 \]
Best Invertibility Test

Current Invertibility Tests

For square $A$, TFAE

1. $A$ represents an invertible process
2. there is a matrix $B$ (same size as $A$) with $AB = I_n = BA$
3. for every $b$, if $Ax = b$ has a solution it is unique
4. $Ax = 0$ has only one solution ($x = 0$)
5. for every $b$, $Ax = b$ has at least one solution

Crucial Facts

1. If $Ax = b$ has a solution, the number is independent of $b$
2. $Ax = 0$ has a solution ($x = 0$)
Computing the Inverse

Strategy

Solve $Ax = e_j$ simultaneously!

\[
\begin{bmatrix}
1 & 2 & 3 & | & 1 & 0 & 0 \\
2 & 3 & 4 & | & 0 & 1 & 0 \\
3 & 4 & 6 & | & 0 & 0 & 1 \\
\end{bmatrix}
\]
Check Your Answer

\[
\begin{bmatrix}
-2 & 0 & 1 \\
0 & 3 & -2 \\
1 & -2 & 1
\end{bmatrix}
\]

is the inverse for \( A \).

Check: \( AB = I_3 \) and \( BA = I_3 \)

Summary

1. To answer “Is \( A \) invertible?”, look for solutions to \( Ax = \theta \).
2. To answer “What is the inverse of \( A \)?”, apply full Gaussian Elimination to \( [A \ I_n] \).
Check Your Answer

\[
\begin{bmatrix}
-2 & 0 & 1 \\
0 & 3 & -2 \\
1 & -2 & 1
\end{bmatrix}
\]
is the inverse for \( A \).

Check: \( AB = I_3 \) and \( BA = I_3 \)

Summary

1. To answer “Is \( A \) invertible?”, look for solutions to \( Ax = b \).
2. To answer “What is the inverse of \( A \)?”, apply full Gaussian Elimination to \( [A \ I_n] \)

Note: Can do this without knowing that \( A \) is invertible.
A square matrix is invertible if

1. \( A \) represents an invertible process
2. there is a matrix \( B \) (same size as \( A \)) with \( AB = I_n \)
3. there is a matrix \( B \) (same size as \( A \)) with \( BA = I_n \)
4. for every \( b \), if \( Ax = b \) has a solution it is unique
5. \( Ax = 0 \) has only one solution (\( x = 0 \))
6. for every \( b \), \( Ax = b \) has at least one solution
7. \( \text{det } A \neq 0 \)
\( \hat{A}x = \hat{b} \)  
\( \frac{\frac{3}{3}\hat{G}E}{V} \)  
\( \frac{\frac{3}{3}\hat{G}E}{V} \)  
\( \frac{\frac{3}{3}\hat{B}x}{V} \)  
\( \frac{\frac{3}{3}\hat{B}x}{V} \)  
\( B\hat{A}x = \hat{B}b \)  

\( 2x + 3y = u \)  
\( 3x + 4y = v \)  
\( v = 3 \)  
\( u = 4 \)  
\( \text{matrix} \)  
\( \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \)  

\( 3x + 4y = 3 \)  
\( 2x + 3y = 4 \)
\[2x + 3y = u\]
\[3x + 4y = v\]

\[2u = 4\]
\[v = 3\]

\[4x + 6y = -4\]
\[3x + 4y = 3\]
\[2x + 3y = u\]
\[3x + 4y = v\]

\[u = 3\]
\[u + v = 4\]

\[2x + 3y = 3\]
\[5x + 7y = 4\]

\[
\begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}, \ 
\begin{bmatrix}
2 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \ 
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \ 
\begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}, \ 
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}
\]