TMA4115 Matematikk 3

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Lecture 2: Complex Numbers and Powers

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Aims and Objectives

By the end of this lecture, you will

- have seen how powers work with complex numbers
- have seen a definition of the complex exponential function
Recap: Complex Numbers

Complex Numbers

\( \mathbb{C} \)

\( i : i^2 = -1 \)

\( z \longleftrightarrow x + iy \)
Recap: Addition, Multiplication, and Division

\[ \mathbb{C} \quad \text{Cartesian} \]

\[ z \quad x + iy \]

Add: \[ z + w \quad (x + u) + i(y + v) \]

Multiply: \[ zw \quad (xu - yv) + i(xv + yu) \]

Reciprocal: \[ z^{-1} \quad \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2} \]
Recap: Polar Coordinates and the Complex Plane

\[ z \leftrightarrow x + iy \leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} \leftrightarrow (r, \theta) \]
Taking Powers

The Next Step

1. Addition: $2 + 3 = 5$

Question: Can we do this with complex numbers?
Taking Powers

The Next Step

1. Addition: \( 2 + 3 = 5 \)
2. Repeated Addition: \( 2 + 2 + 2 = 6 \)

Question: Can we do this with complex numbers?
Taking Powers

The Next Step

1. Addition: $2 + 3 = 5$

2. Repeated Addition: $2 + 2 + 2 = 6$
   = Multiplication: $2 \times 3 = 6$

Question: Can we do this with complex numbers?
Taking Powers

The Next Step

1. Addition: \( 2 + 3 = 5 \)
2. Repeated Addition: \( 2 + 2 + 2 = 6 \)  
   = Multiplication: \( 2 \times 3 = 6 \)
3. Extend: \( e \times \pi = 8.5397 \ldots \)

Question.

Can we do this with complex numbers?
Taking Powers

The Next Step

1. Addition: \( 2 + 3 = 5 \)
2. Repeated Addition: \( 2 + 2 + 2 = 6 \)
   = Multiplication: \( 2 \times 3 = 6 \)
3. Extend: \( e \times \pi = 8.5397 \ldots \)
4. Repeated Multiplication: \( 2 \times 2 \times 2 = 8 \)

Question: Can we do this with complex numbers?
Taking Powers

The Next Step

1. Addition: \(2 + 3 = 5\)
2. Repeated Addition: \(2 + 2 + 2 = 6\)
   = Multiplication: \(2 \times 3 = 6\)
3. Extend: \(e \times \pi = 8.5397 \ldots\)
4. Repeated Multiplication: \(2 \times 2 \times 2 = 8\)
   = Powers: \(2^3 = 8\)

Question:
Can we do this with complex numbers?
Taking Powers

The Next Step

1. **Addition:** \( 2 + 3 = 5 \)

2. **Repeated Addition:** \( 2 + 2 + 2 = 6 \)
   \( \Rightarrow \text{Multiplication: } 2 \times 3 = 6 \)

3. **Extend:** \( e \times \pi = 8.5397 \ldots \)

4. **Repeated Multiplication:** \( 2 \times 2 \times 2 = 8 \)
   \( \Rightarrow \text{Powers: } 2^3 = 8 \)

5. **Extend:** \( e^\pi = 23.141 \ldots \)

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**Question:** Can we do this with complex numbers?
Taking Powers

The Next Step

1. Addition: $2 + 3 = 5$
2. Repeated Addition: $2 + 2 + 2 = 6$
   \quad =\text{ Multiplication: } 2 \times 3 = 6
3. Extend: $e \times \pi = 8.5397 \ldots$
4. Repeated Multiplication: $2 \times 2 \times 2 = 8$
   \quad =\text{ Powers: } 2^3 = 8
5. Extend: $e^\pi = 23.141 \ldots$

Question
Can we do this with complex numbers?
Powers of Real Numbers

Caution: Not Straightforward

1. Powers not commutative: 
   \[ a + b = b + a, \quad a \times b = b \times a, \quad a^b \neq b^a. \]

2. Not all powers always defined: 
   \[ 0^{-1} = ? \quad (-1)^{1/2} = ? \]

3. Not all powers always unique: 
   \[ 4^{1/2} = \pm 2 \quad (-2)^{1/2} = ? \]
Powers of Real Numbers

Caution: Not Straightforward Even for \( \mathbb{R} \)

1. Powers not commutative:
   \[ a^b \neq b^a \]

2. Not all powers always defined:
   \[ 0^{-1/2} = ? \]

3. Not all powers always unique:
   \[ 1^{1/2} = 2^{-2} \]
Powers of Real Numbers

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Not all powers always defined:

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Not all powers always unique:

\[ 4^{1/2} = \begin{cases} 2 \\ -2 \end{cases} \]
Rules of Powers

1. $a^n$ is repeated multiplication:

$$a^n = a \times a \times \cdots \times a$$

2. Multiplication of powers is addition of exponents:

$$a^b \times a^c = a^{b+c}$$

3. Repeated powers is multiplication of exponents:

$$a^{bc} = a^{b \times c}$$
Reduction to Exponentials

Case I: Exponent is Complex

\[ a^{x+iy} \]
Case I: Exponent is Complex

\[ a^{x+iy} = a^x a^{iy} \]

Rule II

Conclusion: To compute \( a^z \), need to know \( e^{it} \) where \( t \in \mathbb{R} \)
Case I: Exponent is Complex

\[ a^{x+iy} = a^x a^{iy} \]

Rule II

\[ = a^x (e^{\log a})^{iy} \]

Conclusion: To compute \( a^z \) need to know \( e^{it} \) where \( t \in \mathbb{R} \).
Reduction to Exponentials

Case I: Exponent is Complex

\[ a^{x+iy} = a^x a^{iy} \quad \text{Rule II} \]
\[ = a^x (e^{\log a})^{iy} \]
\[ = a^x e^{iy \log a} \quad \text{Rule III} \]
Reduction to Exponentials

Case I: Exponent is Complex

\[ a^{x+iy} = a^x a^{iy} \quad \text{Rule II} \]
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Conclusion: To compute \( a^z \) need to know \( e^{it} \) where \( t \in \mathbb{R} \)
Defining Exponentials

Question

How do we define $e^{it}$ for $t \in \mathbb{R}$?
Defining Exponentials

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How do we define $e^{it}$ for $t \in \mathbb{R}$?

How do we define $e^t$ for $t \in \mathbb{R}$?
Course Description

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**Definition**

For $t \in \mathbb{R}$, $e^t$ is the value at $t$ of the unique solution of the ODE

$$y' = y, \quad y(0) = 0.$$
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Exponentials via Differential Equations

**Definition**

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**Definition**

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z' = iz, \quad y(0) = 0.
\]
# Exponentials via Differential Equations

## Definition

For $t \in \mathbb{R}$, $e^{it}$ is the value at $t$ of the unique solution of the ODE

$$z' = iz, \quad y(0) = 0.$$  

## Definition

For $t \in \mathbb{R}$, $e^{it}$ is the value at 1 of the unique solution of the ODE

$$z' = itz, \quad y(0) = 0.$$
In Pictures
Step size: 0.5
Step size: 0.4
Step size: 0.3
Step size: 0.2
Step size: 0.1
Step size: 0.01
In Symbols

To Solve

\[ z' = iz, \quad z(0) = 1, \quad z(t) \in \mathbb{C} \]
In Symbols

To Solve

\[ z' = iz, \quad z(0) = 1, \quad z(t) \in \mathbb{C} \]

1. Write \( z(t) = x(t) + iy(t) \)
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\[ z' = iz, \quad z(0) = 1, \quad z(t) \in \mathbb{C} \]

1. Write \( z(t) = x(t) + iy(t) \)
2. ODE becomes \( x'(t) + iy'(t) = i(x(t) + iy(t)) = -y(t) + ix(t) \)
To Solve

\[ z' = iz, \quad z(0) = 1, \quad z(t) \in \mathbb{C} \]

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3. Separate: \( x'(t) = -y(t), \quad y'(t) = x(t) \)
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3. Separate: \( x'(t) = -y(t), \quad y'(t) = x(t) \)
4. Separate further: \( x''(t) = -x(t) \),
In Symbols

To Solve

\[ z' = iz, \quad z(0) = 1, \quad z(t) \in \mathbb{C} \]

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2. ODE becomes \( x'(t) + iy'(t) = i(x(t) + iy(t)) = -y(t) + ix(t) \)
3. Separate: \( x'(t) = -y(t), \quad y'(t) = x(t) \)
4. Separate further: \( x''(t) = -x(t), \)
5. Solutions: \( x(t) = a \cos(t) + b \sin(t), \)
In Symbols

To Solve

\[ z' = iz, \quad z(0) = 1, \quad z(t) \in \mathbb{C} \]

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5. Solutions: \( x(t) = a \cos(t) + b \sin(t), \)
6. Initial conditions: \( x(0) = 1 \) so \( x(t) = \cos(t) \)
In Symbols

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6. Initial conditions: \( x(0) = 1 \) so \( x(t) = \cos(t) \)
7. Substitute: \( y(t) = -x'(t) = \sin(t) \)
To Solve

\[ z' = iz, \quad z(0) = 1, \quad z(t) \in \mathbb{C} \]

1. Write \( z(t) = x(t) + iy(t) \)
2. ODE becomes \( x'(t) + iy'(t) = i(x(t) + iy(t)) = -y(t) + ix(t) \)
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6. Initial conditions: \( x(0) = 1 \) so \( x(t) = \cos(t) \)
7. Substitute: \( y(t) = -x'(t) = \sin(t) \)
8. Final solution: \( z(t) = \cos(t) + i \sin(t) \)
Truth By Definition

**Definition**

\[ e^{it} = \cos(t) + i \sin(t) \]
Truth By Definition

Definition

\[ e^{it} = \cos(t) + i \sin(t) \]

\[ e^{i\pi} = \cos(\pi) + i \sin(\pi) = -1 \]
Truth By Definition

**Definition**

\[ e^{it} = \cos(t) + i \sin(t) \]

\[ e^{i\pi} = \cos(\pi) + i \sin(\pi) = -1 \]

**Notation:** Polar coordinates

\[ (r, \theta) \leftrightarrow r \cos(\theta) + ir \sin(\theta) = re^{i\theta} \]
Powers

Solved:

\[ a^{x+iy} = a^x e^{iy \log a} \]
Solved:

\[ a^{x+iy} = a^x e^{iy \log a} \]

**Question**

What about \((x + iy)^a\)? Or \((x + iy)^{u+iv}\)?
Case II: Complex Base, Integral Exponent

\[ n \in \mathbb{Z}, \quad (x + iy)^n = ? \]
Case II: Complex Base, Integral Exponent

\[ n \in \mathbb{Z}, \quad (x + iy)^n = ? \]

\[ n \in \mathbb{N}, (x + iy)^n = (x + iy) \times (x + iy) \times \cdots \times (x + iy) \]
Complex Base

Case II: Complex Base, Integral Exponent

\[ n \in \mathbb{Z}, \quad (x + iy)^n = ? \]

\[ n \in \mathbb{N}, (x + iy)^n = (x + iy) \times (x + iy) \times \cdots \times (x + iy) \]

Easier in polar coordinates:

\[ (r, \theta)^n \leftrightarrow (re^{i\theta})^n = r^n e^{in\theta} \]
Case II: Complex Base, Integral Exponent

\[ n \in \mathbb{Z}, \quad (x + iy)^n = ? \]

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Easier in polar coordinates:

\[ (r, \theta)^n \longleftrightarrow (re^{i\theta})^n = r^n e^{in\theta} \]

Then to extend to \( n \in \mathbb{Z} \):

\[ (x + iy)^{-n} = \frac{1}{(x + iy)^n} \]
Case III: Complex Base, Rational Exponent

\[ q \in \mathbb{Q}, \quad (x + iy)^q = ? \]
Complex Base

Case III: Complex Base, Rational Exponent

\[ q \in \mathbb{Q}, \quad (x + iy)^q = ? \]

\[ n \in \mathbb{N}, (x + iy)^{1/n} = ? \]
Case III: Complex Base, Rational Exponent

\[ q \in \mathbb{Q}, \quad (x + iy)^q = ? \]

\[ n \in \mathbb{N}, (x + iy)^{\frac{1}{n}} = ? \]

Solution of:

\[ ((x + iy)^{\frac{1}{n}})^n = x + iy \]
Case III: Complex Base, Rational Exponent

\( q \in \mathbb{Q}, \ (x + iy)^q = ? \)

\( n \in \mathbb{N}, (x + iy)^{1/n} = ? \)

Solution of:

\( \left( (x + iy)^{1/n} \right)^n = x + iy \)

Easier in polar coordinates:

\( s^n e^{in\phi} = re^{i\theta} \)

So \( s^n = r \) and \( n\phi = \theta \).
Complex Base

\[(re^{i\theta})^{\frac{1}{n}} = r^{\frac{1}{n}}e^{i\frac{\theta}{n}}\]
\[ (re^{i\theta})^{\frac{1}{n}} = r^{\frac{1}{n}} e^{i\frac{\theta}{n}} \]

Ambiguous!
Complex Base

\[(re^{i\theta})^{\frac{1}{n}} = r^{\frac{1}{n}}e^{i\frac{\theta}{n}}\]

Ambiguous!

Not \(r^{\frac{1}{n}}\): \(r\) is a length, so positive real number
Complex Base

\[(re^{i\theta})^{1/n} = r^{1/n}e^{i\theta/n}\]

Ambiguous!

Not \(r^{1/n}\): \(r\) is a length, so positive real number

\[e^{2\pi i} = \cos(2\pi) + i\sin(2\pi) = 1\] so \(re^{i\theta} = re^{i\theta+2\pi i}\)

\[(re^{i\theta})^{1/n} = (re^{i\theta+2\pi i})^{1/n} = r^{1/n}e^{i\theta/n+2\pi i/n}\]
Complex Base

\[(re^{i\theta})^{1/n} = r^{1/n}e^{i\theta/n}\]

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Not \(r^{1/n}\): \(r\) is a length, so positive real number

\[e^{2\pi i} = \cos(2\pi) + i\sin(2\pi) = 1\] so \(re^{i\theta} = re^{i\theta+2\pi i}\)

\[(re^{i\theta})^{1/n} = (re^{i\theta+2\pi i})^{1/n} = r^{1/n}e^{i\theta/n+2\pi i/n}\]

... and for \(4\pi i\), \(6\pi i\), ...
Complex Base

\[(re^{i\theta})^{\frac{1}{n}} = r^{\frac{1}{n}}e^{i\frac{\theta}{n}}\]

Ambiguous!

Not \(r^{\frac{1}{n}}\): \(r\) is a length, so positive real number

\[e^{2\pi i} = \cos(2\pi) + i \sin(2\pi) = 1 \text{ so } re^{i\theta} = re^{i\theta + 2\pi i}\]

\[(re^{i\theta})^{\frac{1}{n}} = (re^{i\theta + 2\pi i})^{\frac{1}{n}} = r^{\frac{1}{n}}e^{i\frac{\theta}{n} + \frac{2\pi i}{n}}\]

... and for \(4\pi i, 6\pi i, \ldots\)

Conclusion:

\[(re^{i\theta})^{\frac{1}{n}} = r^{\frac{1}{n}}e^{i\frac{\theta}{n} + \frac{2k\pi i}{n}}, \quad k \in \{0, \ldots, n - 1\}\]
Example: 4th Roots of $4 + i3$

\[
r = \sqrt{4^2 + 3^2} = 5, \quad \theta = \tan^{-1}\left(\frac{4}{3}\right) = 0.9273
\]

\[
\implies 5^{\frac{1}{4}} = 1.4953, \quad \implies 0.23182
\]
Example: 4th Roots of $4 + i3$

\[ r = \sqrt{4^2 + 3^2} = 5, \quad \theta = \tan^{-1}\left(\frac{4}{3}\right) = 0.9273 \]

\[ \Leftrightarrow 5^{1/4} = 1.4953, \quad \Leftrightarrow 0.23182 \]
Example: 4th Roots of $4 + i3$

$$r = \sqrt{4^2 + 3^2} = 5,$$

$$\Rightarrow 5^{\frac{1}{4}} = 1.4953,$$

$$\theta = \tan^{-1}\left(\frac{4}{3}\right) = 0.9273$$

$$\Rightarrow 0.23182 + n\pi/2$$
Powers

In General:

\[(x + iy)^{u+iv} = (re^{i\theta})^{u+iv}\]
Powers

In General:

\[(x + iy)^{u+iv} = (re^{i\theta})^{u+iv} = r^u r^{iv} e^{i\theta u} e^{i\theta v}\]
Powers

In General:

\[(x + iy)^{u+iv} = (re^{i\theta})^{u+iv} = r^u r^{iv} e^{i\theta u} e^{i\theta iv} = r^u e^{-\theta v} e^{i(v \log(r) + \theta u)}\]
In General:

\[(x + iy)^{u+iv} = (re^{i\theta})^{u+iv} = r^u r^i v e^{i\theta u} e^{i\theta v} = r^u e^{-\theta v} e^{i(v \log(r)+\theta u)}\]

Same ambiguity: \(x + iy = re^{i\theta + 2k\pi i}\), but can be infinitely many solutions
Powers

In General:

\[(x + iy)^{u+iv} = (re^{i\theta})^{u+iv} = r^u r^{iv} e^{i\theta u} e^{i\theta iv} = r^u e^{-\theta v} e^{i(v \log(r) + \theta u)}\]

Same ambiguity: \(x + iy = re^{i\theta + 2k\pi i}\), but can be infinitely many solutions

**Neat Fact**

\[i^i\]
Powers

In General:

\[(x + iy)^{u+iv} = (re^{i\theta})^{u+iv}\]
\[= r^u r^i e^{i\theta u} e^{i\theta iv}\]
\[= r^u e^{-\theta v} e^{i(v \log(r)+\theta u)}\]

Same ambiguity: \(x + iy = re^{i\theta+2k\pi i}\), but can be infinitely many solutions

Neat Fact

\[i^i = \left(e^{\pi/2}\right)^i\]
Powers

In General:

\[(x + iy)^{u + iv} = (re^{i\theta})^{u + iv} \]

\[= r^u r^{iv} e^{i\theta u} e^{i\theta iv} \]

\[= r^u e^{-\theta v} e^{i(\nu \log(r) + \theta u)} \]

Same ambiguity: \( x + iy = re^{i\theta + 2k\pi i} \), but can be infinitely many solutions

Neat Fact

\[i^i = (e^{i\pi/2})^i = e^{i^2\pi/2} \approx 0.2078 \approx 1/5 \]
Powers

In General:

\[(x + iy)^{u+iv} = (re^{i\theta})^{u+iv} = r^u r^{iv} e^{i\theta u} e^{i\theta iv} = r^u e^{-\theta v} e^{i(v \log(r) + \theta u)}\]

Same ambiguity: \(x + iy = re^{i\theta + 2k\pi i}\), but can be infinitely many solutions

Neat Fact

\[i^i = \left(e^{i\pi/2}\right)^i = e^{i2\pi/2} = e^{-\pi/2}\]
Powers

In General:

\[(x + iy)^{u+iv} = (re^{i\theta})^{u+iv}\]
\[= r^u r^{iv} e^{i\theta u} e^{i\theta v}\]
\[= r^u e^{-\theta v} e^{i(\nu \log(r) + \theta u)}\]

Same ambiguity: \(x + iy = re^{i\theta + 2k\pi i}\), but can be infinitely many solutions

Neat Fact

\[i^i = (e^{i\pi/2})^i = e^{i^2 \pi/2} = e^{-\pi/2} \approx 0.20788 \approx 1/5\]
Pretty Pictures

 Addition, Subtraction
Pretty Pictures

Addition, Subtraction
Pretty Pictures

Addition, Subtraction
Complex conjugation, real and imaginary parts
Complex conjugation, real and imaginary parts
Pretty Pictures

Complex conjugation, real and imaginary parts
Pretty Pictures

Complex conjugation, real and imaginary parts
Pretty Pictures

Polar Co-ordinates
Pretty Pictures

Polar Co-ordinates
Pretty Pictures

Polar Co-ordinates

\( e^{0.64350} \)
Pretty Pictures

Polar Co-ordinates

\[ \sin(0.64350) \]

\[ \cos(0.64350) \]
Pretty Pictures

Absolute Value: \( z = x + iy \mapsto |z| = \sqrt{x^2 + y^2} \)

Argument: \( z = x + iy \mapsto \arg(z) = \tan^{-1}(y/x) \)
\[
\sqrt{13} = \sqrt{3^2 + 2^2}
\]

**Absolute Value:** 
If \( z = x + iy \), then 
\[
|z| = \sqrt{x^2 + y^2}
\]

**Argument:** 
If \( z = x + iy \), then 
\[
\arg(z) = \tan^{-1}(\frac{y}{x})
\]
Multiplication = Rotation and Dilation
Multiplication = Rotation and Dilation
Pretty Pictures

Multiplication = Rotation and Dilation
Reciprocal: $z \mapsto \frac{1}{z}$
Not Just Pretty Pictures

Reciprocal: $z \mapsto \frac{1}{z}$
Reciprocal: \( z \mapsto \frac{1}{z} \)
Not Just Pretty Pictures
Not Just Pretty Pictures

\[ z \mapsto e^z \]
Not Just Pretty Pictures

\[ z \mapsto e^z \]
Not Just Pretty Pictures

\[ z \mapsto e^{z} \]
Write the complex number \( w = \frac{3-i}{2i-1} \) in polar form. Find all the solutions of the equation \( z^4 = w \) and draw the solutions in the complex plane.

\[
w = \sqrt{2}e^{i\frac{5\pi}{4}}
\]
Question

Write the complex number $w = \frac{3-i}{2i-1}$ in polar form. Find all the solutions of the equation $z^4 = w$ and draw the solutions in the complex plane.

\[ w = \sqrt{2}e^{i\pi/4} \]

\[ z = re^{i\theta} \]

\[ \Rightarrow \quad r^4 = \sqrt{2}, \]

\[ 4\theta = \frac{5\pi}{4} \]

\[ \Rightarrow \quad r = 2^{1/8}, \]

\[ \theta = \frac{5\pi}{16} \]
Question
Write the complex number \( w = \frac{3-i}{2i-1} \) in polar form. Find all the solutions of the equation \( z^4 = w \) and draw the solutions in the complex plane.

\[
\begin{align*}
w &= \sqrt{2}e^{i\pi/4} \\
z &= re^{i\theta} \\

\implies r^4 &= \sqrt{2}, \\
\quad 4\theta &= \frac{5\pi}{4} \\

\implies r &= 2^{1/8}, \\
\theta &= \frac{5\pi}{16}(\pi/2)
\end{align*}
\]
Write the complex number $w = \frac{3-i}{2i-1}$ in polar form. Find all the solutions of the equation $z^4 = w$ and draw the solutions in the complex plane.

\[ w = \sqrt{2}e^{i\pi/4} \]
\[ z = re^{i\theta} \]

\[ \implies r^4 = \sqrt{2}, \]
\[ 4\theta = \frac{5\pi}{4} \]

\[ \implies r = 2^{1/8}, \]
\[ \theta = \frac{5\pi}{16}(+\pi/2(+\pi/2(+\pi/2))) \]
On the Complex Plane
Summary

- Powers of complex numbers by complex numbers defineable
- Usually multi-valued
- Formula for roots (polar coordinates):
  \[ (r e^{i\theta})^{1/n} = r^{1/n} e^{i\theta/n + i2\pi/n} \]
- For \( w \neq 0 \), \( z^n = w \) has exactly \( n \) distinct solutions
- \( e^{it} \) is unique solutions of \( z'(t) = z(t) \) with initial condition \( z(0) = 1 \)
- \( e^{it} = \cos(t) + i \sin(t) \)