TMA4115 Matematikk 3

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Lecture 9: Linear Systems

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By the end of this lecture, you will
- know what a linear system is
- know what it means to solve one
- have seen the idea of how to solve one
Recap

2nd Order Linear ODE

\[ y'' + p(t)y' + q(t)y = r(t) \]

Method

1. Solve homogenous in full
   - Reduction of order
   - Constant coefficients
   - Euler–Cauchy

2. Find particular solution
   - Guess (intelligently)
   - Variation of parameters

3. Fit to initial conditions
Recap: Initial Conditions

Example

\[ y'' - y = 2, \quad y(0) = 0, \quad y'(0) = 4 \]

General Solution: \[ y(t) = Ae^t + Be^{-t} + 2 \]
Recap: Initial Conditions

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General Solution: \( y(t) = Ae^t + Be^{-t} + 2 \)

Fit to Initial Conditions: Find \( A \) and \( B \) such that:

\[
\begin{align*}
A + B + 2 &= 0 \\
A - B &= 4
\end{align*}
\]

Always Check!
Recap: Initial Conditions

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Solution: \( A = 1, \ B = -3 \)
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Example

\[ y'' - y = 2, \quad y(0) = 0, \quad y'(0) = 4 \]

General Solution: \[ y(t) = Ae^t + Be^{-t} + 2 \]

Fit to Initial Conditions: Find \( A \) and \( B \) such that:

\[
\begin{align*}
1 + &- 3 + 2 = 0 \\
1 - &- 3 = 4
\end{align*}
\]

Solution: \( A = 1, \ B = -3 \)

Always Check!
Recap: Initial Conditions

Example

\[ y'' - y = 2, \quad y(0) = 0, \; y'(0) = 4 \]

General Solution: \( y(t) = Ae^t + Be^{-t} + 2 \)

Fit to Initial Conditions: Find \( A \) and \( B \) such that:

\[
\begin{align*}
1 + A - 3B + 2 &= 0 \quad \checkmark \\
1 - A - 3 &= 4 \quad \checkmark
\end{align*}
\]

Solution: \( A = 1, \; B = -3 \)

Always Check!
Find $x_1, x_2, x_3, x_4 \in \mathbb{R}$ such that

\[-2x_1 + 6x_2 + 4x_3 + 4x_4 = 0\]
\[x_1 - 2x_3 + 6x_4 = -5\]
\[-3x_1 + 3x_2 + 6x_3 = 0\]
Themes from ODEs: Linearity

Compare and Contrast

2nd Order Linear ODE

\[ y'' + p(t)y' + q(t)y = r(t) \]

Exam Question

\[-2x_1 + 6x_2 + 4x_3 + 4x_4 = 0 \]
\[ x_1 - 2x_3 + 6x_4 = -5 \]
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**Compare and Contrast**

*2nd Order Linear ODE*

\[ y'' + p(t)y' + q(t)y = r(t) \]

*Exam Question*

\[-2x_1 + 6x_2 + 4x_3 + 4x_4 = 0 \]
\[ x_1 - 2x_3 + 6x_4 = -5 \]
\[-3x_1 + 3x_2 + 6x_3 = 0 \]

**Common Features**

- Each term is “pure”: \( x_1 \) not \( x_1^2 \)
- No interactions: \(-2x_1 + 6x_2 \) not \( x_1x_2 \)
Themes from ODEs: Focus on Solutions

## Compare and Contrast

**2nd Order Linear ODE:** Find \( y : I \rightarrow \mathbb{R} \) such that

\[
y'' + p(t)y' + q(t)y = r(t)
\]

**Exam Question:** Find \( x_1, x_2, x_3, x_4 \in \mathbb{R} \) such that

\[
\begin{align*}
-2x_1 + 6x_2 + 4x_3 + 4x_4 &= 0 \\
x_1 - 2x_3 + 6x_4 &= -5 \\
-3x_1 + 3x_2 + 6x_3 &= 0
\end{align*}
\]
Compare and Contrast

2nd Order Linear ODE: Find $y : I \rightarrow \mathbb{R}$ such that

$$y'' + p(t)y' + q(t)y = r(t)$$

Exam Question: Find $x_1, x_2, x_3, x_4 \in \mathbb{R}$ such that

$$-2x_1 + 6x_2 + 4x_3 + 4x_4 = 0$$
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$$-3x_1 + 3x_2 + 6x_3 = 0$$

Common Features

The solution is primary, the problem and the method are secondary.
Linear System as Linear Process

1. I put in $X$, what do I get?

2. I got $Y$, where did I start?

3. I put in $X$, I got $Y$, how did I get it?
Linear System as Linear Process

Input ➔ Process ➔ Output

Question

1. I put in $X$, what do I get?
Linear System as Linear Process

**Question**

1. I put in $X$, what do I get?
2. I got $Y$, where did I start?
Linear System as Linear Process

Input → Process → Output

Question:
1. I put in $X$, what do I get?
2. I got $Y$, where did I start?
3. I put in $X$, I got $Y$, how did I get it?
Imagine

$$X_1 + X_2$$ makes sense for inputs
$$Y_1 + Y_2$$ makes sense for outputs
Imagine

\[ X_1 + X_2 \text{ makes sense for inputs} \]
\[ Y_1 + Y_2 \text{ makes sense for outputs} \]

If

\[ X_1 \leftrightarrow Y_1 \]
and

\[ X_2 \leftrightarrow Y_2 \]
then

\[ X_1 + X_2 \leftrightarrow Y_1 + Y_2 \]
### Imagine

- $X_1 + X_2$ makes sense for inputs
- $Y_1 + Y_2$ makes sense for outputs

If

- $X_1 \mapsto Y_1$
  - and
- $X_2 \mapsto Y_2$
  - then
- $X_1 + X_2 \mapsto Y_1 + Y_2$

### Key Idea

Knowing that $X_1 \mapsto Y_1$ and $X_2 \mapsto Y_2$ means knowing what $X_1 + X_2$ goes to without having to compute it.
Grandiose Claim

Any answerable question can be turned into a question about numbers.
Grandiose Claim

Any answerable question can be turned into a question about numbers.

Key Point

(Almost) any information can be represented by numbers.
Grandiose Claim
Any answerable question can be turned into a question about numbers.

Key Point
(Almost) any information can be represented by numbers.

But different representations lead to different sets of numbers!
## Imagine

Two representations:

1. One simplest for **description**
2. One simplest for **prescription**
Imagine

Two representations:

1. One simplest for description
2. One simplest for prescription

Key Question

Suppose we have a description, how do we turn it into a prescription?
Imagine

Two representations:

1. One simplest for description
2. One simplest for prescription

Key Question

Suppose we have a description, how do we turn it into a prescription?

How do we translate that description to the other way?
Definition of a Linear System

( Easier said than done! )
Definition of a Linear System

( Easier said than done! )

Definition

A linear system is a finite list of equations in a finite number of unknown variables, $x_1, \ldots, x_n$, of the form

$$
\begin{align*}
  a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &= b_1 \\
  a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &= b_2 \\
  \vdots & \quad \vdots \quad \vdots \quad \vdots \\
  a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &= b_m
\end{align*}
$$

where $a_{i,j}, b_j \in \mathbb{R}$.
Definition of a Linear System

(Easier said than done!)

Definition

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 &\vdots & \vdots & \vdots & \vdots \\
 a_{m,1} x_1 + a_{m,2} x_2 + \cdots + a_{m,n} x_n &= b_m
\end{align*}
\]

where \( a_{i,j}, b_j \in \mathbb{R} \).

(Now forget this definition!)
A **solution** of a linear system is a choice of numbers $x_1, \ldots, x_n$ which make all the equations *true*. That is, $x_1, \ldots, x_n \in \mathbb{R}$ such that

\[
\begin{align*}
    a_{1,1} x_1 + a_{1,2} x_2 + \cdots + a_{1,n} x_n &= b_1 \\
    a_{2,1} x_1 + a_{2,2} x_2 + \cdots + a_{2,n} x_n &= b_2 \\
    \vdots & \quad \vdots \quad \vdots \quad \vdots \\
    a_{m,1} x_1 + a_{m,2} x_2 + \cdots + a_{m,n} x_n &= b_m
\end{align*}
\]

simultaneously!
Examples of Linear Systems

1

\[
x_1 + x_2 = -2 \\
x_1 - x_2 = 0
\]
Examples of Linear Systems

1

\[ x_1 + x_2 = -2 \]
\[ x_1 - x_2 = 0 \]

Solution: \( x_1 = 1, \ x_2 = -3 \)
Examples of Linear Systems

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\[ \begin{align*}
    x_1 + x_2 &= -2 \\
    x_1 - x_2 &= 0
\end{align*} \]

Solution: \( x_1 = 1, \ x_2 = -3 \)

2

\[ \begin{align*}
    -2x_1 + 6x_2 + 4x_3 + 4x_4 &= 0 \\
    x_1 - 2x_3 + 6x_4 &= -5 \\
    -3x_1 + 3x_2 + 6x_3 &= 0
\end{align*} \]
Examples of Linear Systems

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\[
\begin{align*}
  x_1 + x_2 &= -2 \\
  x_1 - x_2 &= 0
\end{align*}
\]

Solution: \( x_1 = 1, \ x_2 = -3 \)

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\begin{align*}
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  x_1 - 2x_3 + 6x_4 &= -5 \\
  -3x_1 + 3x_2 + 6x_3 &= 0
\end{align*}
\]

Solution: \( x_1 = 1, \ x_2 = 1, \ x_3 = 0, \ x_4 = -1 \)
Examples of Linear Systems

1

\[ x_1 + x_2 = -2 \]
\[ x_1 - x_2 = 0 \]

Solution: \( x_1 = 1, \ x_2 = -3 \)

2

\[ -2x_1 + 6x_2 + 4x_3 + 4x_4 = 0 \]
\[ x_1 - 2x_3 + 6x_4 = -5 \]
\[ -3x_1 + 3x_2 + 6x_3 = 0 \]

Solution: \( x_1 = 1, \ x_2 = 1, \ x_3 = 0, \ x_4 = -1 \)

3

\[ 2x_1 + 3x_2 + 4x_3 = 4 \]
\[ x_2 + x_3 = 1 \]
\[ x_1 - 4x_2 - x_3 = 4 \]
Examples of Linear Systems

1

\[
\begin{align*}
  x_1 + x_2 &= -2 \\
  x_1 - x_2 &= 0
\end{align*}
\]

Solution: \( x_1 = 1, \ x_2 = -3 \)

2

\[
\begin{align*}
  -2x_1 + 6x_2 + 4x_3 + 4x_4 &= 0 \\
  x_1 - 2x_3 + 6x_4 &= -5 \\
  -3x_1 + 3x_2 + 6x_3 &= 0
\end{align*}
\]

Solution: \( x_1 = 1, \ x_2 = 1, \ x_3 = 0, \ x_4 = -1 \)

3

\[
\begin{align*}
  2x_1 + 3x_2 + 4x_3 &= 4 \\
  x_2 + x_3 &= 1 \\
  x_1 - 4x_2 - x_3 &= 4
\end{align*}
\]

Solution: \( x_1 = -1, \ x_2 = -2, \ x_3 = 3 \)
<table>
<thead>
<tr>
<th>Number</th>
<th>Equations</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(3x + 4y = 2) (y = -1)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>(5x + 7y = 3) (4x + 6y = 2)</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>(3x + 4y = 2) (3y = -3)</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>(3x = 6) (y = -1)</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>(x = 2) (y = -1)</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>(3x + 4y = 2) (2x + 3y = 1)</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>(x + y = 1) (x + 2y = 0)</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>(x = 2) (2x + 3y = 1)</td>
<td></td>
</tr>
</tbody>
</table>
Similar Problems Have Similar Solutions

**Rough Definition**

A linear system is a way of specifying a list of numbers by giving conditions they must satisfy.
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Focus on the solution
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A linear system is a way of specifying a list of numbers by giving conditions they must satisfy.

Focus on the solution

\[\begin{align*}
3x + 4y &= 2 \\
2x + 3y &= 1
\end{align*}\]

and

\[\begin{align*}
x &= 2 \\
y &= -1
\end{align*}\]

have the same solution: \(x = 2, \ y = -1\)
A linear system is a way of specifying a list of numbers by giving conditions they must satisfy.

Focus on the solution

\[3x + 4y = 2\]
\[2x + 3y = 1\]

and

\[x = 2\]
\[y = -1\]

have the same solution: \(x = 2, y = -1\)

Change the given linear system to another one with exactly the same solutions which is easier to solve.
Key Idea

Change the given linear system to another one with exactly the same solutions which is easier to solve.
Elementary Moves

Key Idea

Change the given linear system to another one with exactly the same solutions which is easier to solve.

Initial Thoughts

\[ \begin{align*}
    x &= 2 \\
    y &= -1
\end{align*} \]

is easier than

\[ \begin{align*}
    3x + 4y &= 2 \\
    2x + 3y &= 1
\end{align*} \]

but that’s too far for a single step.
Elementary Moves

Key Idea

Change the given linear system to another one with exactly the same solutions which is easier to solve.

Initial Thoughts

\[ x = 2 \quad \text{is easier than} \quad 3x + 4y = 2 \]
\[ y = -1 \quad \text{is easier than} \quad y = -1 \]
\[ 3x + 4y = 2 \quad \text{is easier than} \quad 2x + 3y = 1 \]
\[ \text{much better, but still too far for a single step.} \]
Elementary Moves

Basic Method

Modify linear system in small steps so that:

1. Each step clearly has exactly the same solutions as the one before
2. Each step is clearly simpler than the one before
Elementary Moves

**Basic Method**

Modify linear system in *small* steps so that:

1. Each step *clearly* has exactly the same solutions as the one before
2. Each step is *clearly* simpler than the one before

**The Moves**

1. **Equation Swap**: swap two equations
2. **Equation Scale**: scale an equation by a *non-zero* number
3. **Add Equations**: add a multiple of one equation to another
Solving a System

\[-2x_1 + 6x_2 + 4x_3 + 4x_4 = 0\]
\[x_1 - 2x_3 + 6x_4 = -5\]
\[-3x_1 + 3x_2 + 6x_3 = 0\]
Solving a System

\[-2x_1 + 6x_2 + 4x_3 + 4x_4 = 0\]
\[x_1 - 2x_3 + 6x_4 = -5\]
\[-3x_1 + 3x_2 + 6x_3 = 0\]
\[x_1 - 3x_2 - 2x_3 - 2x_4 = 0\]
\[x_1 - 2x_3 + 6x_4 = -5\] Scale
\[-3x_1 + 3x_2 + 6x_3 = 0\]
Solving a System

\[ x_1 - 3x_2 - 2x_3 - 2x_4 = 0 \]
\[ x_1 - 2x_3 + 6x_4 = -5 \] Scale
\[ -3x_1 + 3x_2 + 6x_3 = 0 \]
\[ x_1 - 3x_2 - 2x_3 - 2x_4 = 0 \]
\[ 3x_2 + 8x_4 = -5 \] Add
\[ -3x_1 + 3x_2 + 6x_3 = 0 \]
Solving a System

\[ x_1 - 3x_2 - 2x_3 - 2x_4 = 0 \]
\[ 3x_2 + 8x_4 = -5 \quad \text{Add} \]
\[ -3x_1 + 3x_2 + 6x_3 = 0 \]

\[ x_1 - 3x_2 - 2x_3 - 2x_4 = 0 \]
\[ 3x_2 + 8x_4 = -5 \quad \text{Add} \]
\[ -6x_2 - 6x_4 = 0 \]
Solving a System

\[
\begin{align*}
  x_1 - 3x_2 - 2x_3 - 2x_4 &= 0 \\
  3x_2 + 8x_4 &= -5 \quad \text{Add} \\
  -6x_2 - 6x_4 &= 0 \\
  x_1 - 3x_2 - 2x_3 - 2x_4 &= 0 \\
  3x_2 + 8x_4 &= -5 \quad \text{Add} \\
  10x_4 &= -10
\end{align*}
\]
Solving a System

\[ x_1 - 3x_2 - 2x_3 - 2x_4 = 0 \]
\[ 3x_2 + 8x_4 = -5 \quad \text{Add} \]
\[ 10x_4 = -10 \]

\[ x_1 - 3x_2 - 2x_3 - 2x_4 = 0 \]
\[ 3x_2 + 8x_4 = -5 \quad \text{Scale} \]
\[ x_4 = -1 \]
Solving a System

\[
\begin{align*}
    x_1 - 3x_2 - 2x_3 - 2x_4 &= 0 \\
    3x_2 + 8x_4 &= -5 \\
    x_4 &= -1
\end{align*}
\]

Scale

\[
\begin{align*}
    x_1 - 3x_2 - 2x_3 - 2x_4 &= 0 \\
    3x_2 &= 3 \\
    x_4 &= -1
\end{align*}
\]

Add
Solving a System

\[ x_1 - 3x_2 - 2x_3 - 2x_4 = 0 \]
\[ 3x_2 = 3 \quad \text{Add} \]
\[ x_4 = -1 \]
\[ x_1 - 3x_2 - 2x_3 - 2x_4 = 0 \]
\[ x_2 = 1 \quad \text{Scale} \]
\[ x_4 = -1 \]
Solving a System

\[ x_1 - 3x_2 - 2x_3 - 2x_4 = 0 \]
\[ x_2 = 1 \] Scale
\[ x_4 = -1 \]

\[ x_1 - 2x_3 = 1 \]
\[ x_2 = 1 \] Add
\[ x_4 = -1 \]
Solving a System

\[
\begin{align*}
  x_1 - 2x_3 &= 1 \\
  x_2 &= 1 \\
  x_4 &= -1
\end{align*}
\]

Add

\[x_4 = -1, \ x_3 = 0, \ x_2 = 1, \ x_1 = 1\]
Solving a System

\[ x_1 - 2x_3 = 1 \]
\[ x_2 = 1 \quad \text{Add} \]
\[ x_4 = -1 \]

\[ x_4 = -1, \ x_3 = 0, \ x_2 = 1, \ x_1 = 1 \]
But more possible! \[ x_4 = -1, \ x_3 = 1, \ x_2 = 1, \ x_1 = 3 \]
Solving a System

\[ x_1 - 2x_3 = 1 \]
\[ x_2 = 1 \quad \text{Add} \]
\[ x_4 = -1 \]

\[ x_4 = -1, x_3 = 0, x_2 = 1, x_1 = 1 \]

But more possible! \[ x_4 = -1, x_3 = 1, x_2 = 1, x_1 = 3 \]

Have complete freedom on \( x_3 \), then put \( x_1 = 1 + 2x_3 \)
Theorem

If two linear systems are related by an elementary move, they have exactly the same solution set.
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If two linear systems are related by an elementary move, they have exactly the same solution set.

Proof.

Each elementary move is reversible: it can be undone by another elementary move.

1. Swap: swap back
2. Scale: rescale
3. Add a multiple: subtract a multiple (aka add a multiple)

Key point for adding: we keep the one that we added.
Theorem

If two linear systems are related by an elementary move, they have exactly the same solution set.

Proof.

Each elementary move is reversible: it can be undone by another elementary move.

1. Swap: swap back
2. Scale: rescale
3. Add a multiple: subtract a multiple (aka add a multiple)

Key point for adding: we keep the one that we added.

\[
\begin{align*}
x + y &= 3 \\
x - y &= 1
\end{align*}
\]

\[
\begin{align*}
R_2 &\rightarrow R_2 - R_1 \\
x + y &= 3 \\
0 - 2y &= -2 \\
R_2 &\rightarrow R_2 + R_1 \\
x + y &= 3 \\
x - y &= 1
\end{align*}
\]
Loose Ends: Gaussian Elimination

- The Most Important Points:
  1. At each stage, do the obvious.
  2. It works (theoretically).
  3. A computer can do it better.

In more detail:
  1. Goal: use elementary operations to get lots of zeros.
  2. Don't mess with zeros you have already made.
  3. Work systematically to ensure termination.

Most common mistakes:
  1. Work on whole system: LHS and RHS.
  2. Signs: e.g. $-x$.
  3. Computational errors.
Loose Ends: Gaussian Elimination

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2. Signs: e.g. \(-x\)
Loose Ends: Gaussian Elimination

- **The Most Important Points:**
  1. At each stage, do the obvious
  2. It works (theoretically)
  3. A computer can do it better

- **In more detail:**
  1. Goal: use elementary operations to get *lots* of zeros
  2. Don’t mess with zeros *you have already made*
  3. Work systematically to ensure termination

- **Most common mistakes:**
  1. Work on *whole* system: LHS and RHS
  2. Signs: e.g. $-x$
  3. Computational errors
GE: An Algorithm (for a person)

1. Use elementary operations to reduce system
GE: An Algorithm (for a person)

1. Use elementary operations to reduce system
   1. Work from left to right

2. Work from top to bottom

3. Leave alone equations already used

4. Interpret answer correctly

5. Use "back substitution"

6. Use "dummy variables" if have choice
GE: An Algorithm (for a person)

1. Use elementary operations to reduce system
   1. Work from left to right
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Example

\[ 2x_1 + 3x_2 + 4x_3 = 4 \]
\[ x_2 + x_3 = 1 \]
\[ x_1 - 4x_2 - x_3 = 4 \]
Summary

- Linear systems are about **finding solutions**
- Solve linear systems by **simple changes**
  1. Swaps
  2. Scales
  3. Add multiple of one equation to another