TMA4115 Matematikk 3

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Trondheim

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Lecture 26: Quadratic Forms

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Aims and Objectives

By the end of this lecture, you will

- have seen the classification of quadratic forms
- have seen how to relate them to conic sections

Recap: Symmetric Matrices

Definition

Symmetric: $A^\top = A$

Theorem

Symmetric $\iff$ orthogonal basis of eigenvectors

$$f(x+h) \simeq f(x) + \nabla f \bullet h + h \bullet Hfh$$

Typical Exam Question

Question

- Let $A = \begin{bmatrix} 7 & 24 \\ 24 & -7 \end{bmatrix}$. Find the eigenvalues of A and eigenvectors v_1 and v_2 such that the matrix P with column vectors v_1 and v_2 is an orthogonal matrix with determinant 1.
- The equation $7x^2 + 48xy - 7y^2 - 40x - 30y = 0$ describes a conic section in the xy–plane. Find a rotated coordinate system (x', y') where the equation of the conic section is in the form $\lambda_1(x')^2 + \lambda_2(y')^2 + dx' + ey' = 0$. Which type of conic section is it? Draw the new coordinates and the conic section in the xy–plane.

First Part

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Other must be $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$.

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Second Part

- The equation

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describes a conic section in the xy–plane. Find a rotated coordinate system (x', y') where the equation of the conic section is in the form

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Solving an ODE

Planetary Motion:

$$F = ma = -\frac{GMm}{r^2}$$

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Convert to first order:

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} v \\ -\frac{1}{\|u\|^3}u \end{bmatrix}$$

(Note that this is now six dimensional!)

Constants of Motion

Observation: Planets stay on the ecliptic plane.

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Conservation of Energy

Conclusion:

$$\begin{bmatrix} u \\ v \end{bmatrix} \bullet \begin{bmatrix} 0 & \frac{1}{2}I \\ I & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + c \bullet \begin{bmatrix} u \\ v \end{bmatrix} = \text{const}$$

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Not quite symmetric, but almost.

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Basis of Eigenvectors:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ \sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -\sqrt{2} \end{bmatrix} \right\}$$

Eigenvalues: $\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$

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More Generally: ODE $u' = G(u)$ can be difficult to solve so look for conservation laws.

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Then $(u \bullet Au)' = 2u' \bullet Au = 2G(u) \bullet Au = 0$

So $u \bullet Au$ is constant and the solution of $u' = G(u)$ lies on the level set of $u \bullet Au$.

Quadratic Forms

Definition

General Quadratic Form (2D):

$$ax^2 + 2bxy + cy^2 + dx + ey + f = g$$

Matrix Reformulation:

$$\begin{bmatrix} x \\ y \end{bmatrix} \bullet \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} d \\ e \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = g$$

Classification

Procedure:

1. Find orthogonal eigenbasis of $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$

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4. $\begin{bmatrix} d \\ e \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = DX + EY$

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4. $\begin{bmatrix} d \\ e \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = DX + EY$
5. $\lambda_1 X^2 + \lambda_2 Y^2 + DX + EY = \lambda_1 (X - {}^{D}\!/_{2\lambda_1})^2 + \lambda_2 (Y - {}^{E}\!/_{2\lambda_2})^2 - G$

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6. Translate: $\overline{X} = X - {}^{D}\!/_{2\lambda_1}$, $\overline{Y} = Y - {}^{E}\!/_{2\lambda_2}$

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4. $\begin{bmatrix} d \\ e \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = DX + EY$
5. $\lambda_1 X^2 + \lambda_2 Y^2 + DX + EY = \lambda_1 (X - D/2\lambda_1)^2 + \lambda_2 (Y - E/2\lambda_2)^2 - G$
6. Translate: $\overline{X} = X - D/2\lambda_1$, $\overline{Y} = Y - E/2\lambda_2$
7. Original system is equivalent to $\lambda_1 \overline{X}^2 + \lambda_2 \overline{Y}^2 = G$, but rotated and translated

Classification

Possibilities:

1. $x^2 + 2y^2 = 1$
2. $x^2 + 2y^2 = 0$
3. $x^2 + 2y^2 = -1$

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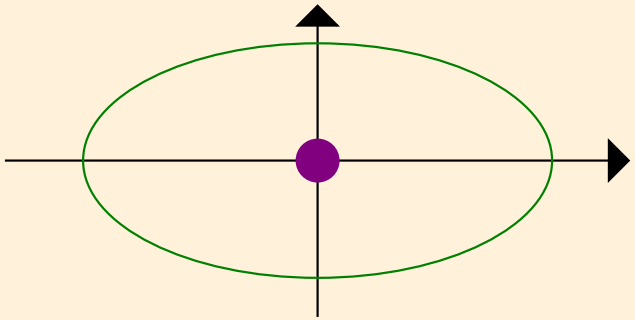
1. $x^2 + 2y^2 = 1$ Ellipse
2. $x^2 + 2y^2 = 0$ Point
3. $x^2 + 2y^2 = -1$ No Solutions



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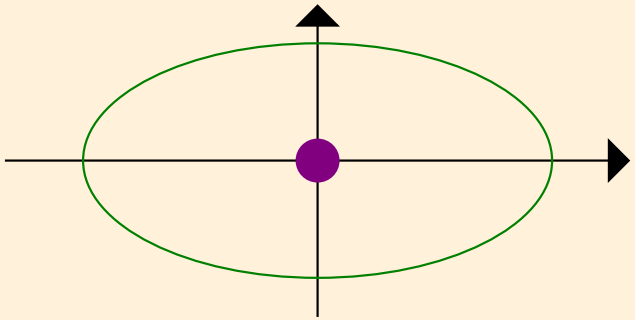

4. $x^2 - 2y^2 = 1$
5. $x^2 - 2y^2 = 0$

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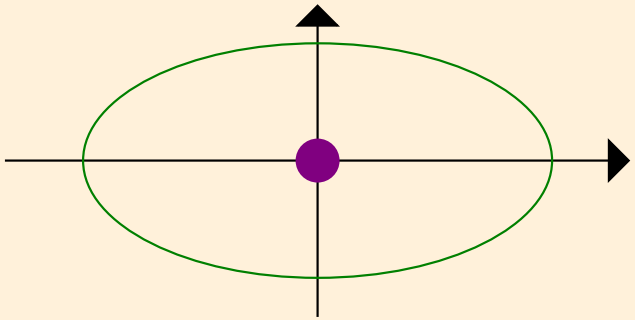
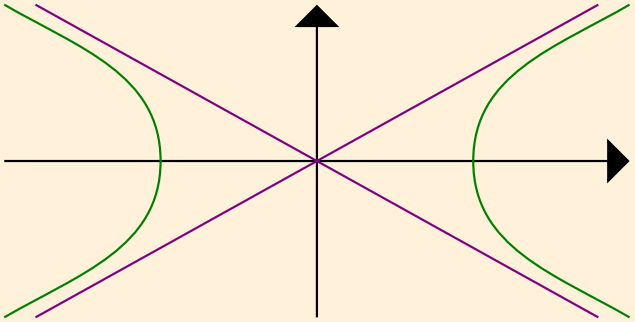


4. $x^2 - 2y^2 = 1$ Hyperbola
5. $x^2 - 2y^2 = 0$ Lines



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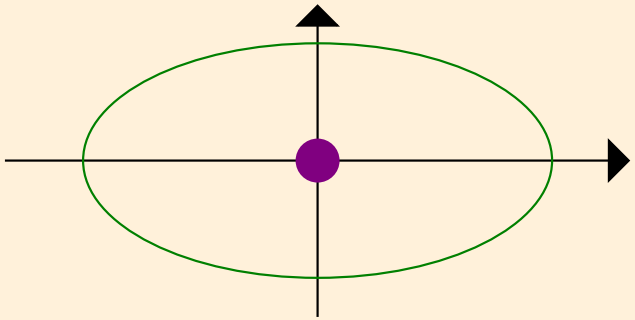
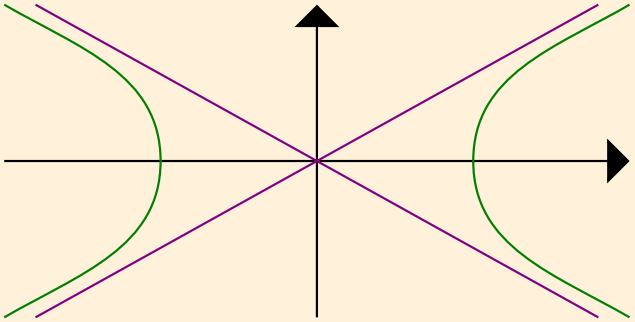
2. $x^2 + 2y^2 = 0$ Point
3. $x^2 + 2y^2 = -1$ No Solutions



4. $x^2 - 2y^2 = 1$ Hyperbola
5. $x^2 - 2y^2 = 0$ Lines

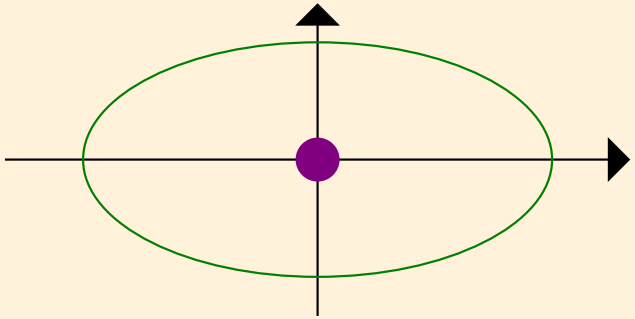


6. $x^2 - 2y = 1$

Classification

Possibilities:

1. $x^2 + 2y^2 = 1$ Ellipse
2. $x^2 + 2y^2 = 0$ Point
3. $x^2 + 2y^2 = -1$ No Solutions



4. $x^2 - 2y^2 = 1$ Hyperbola
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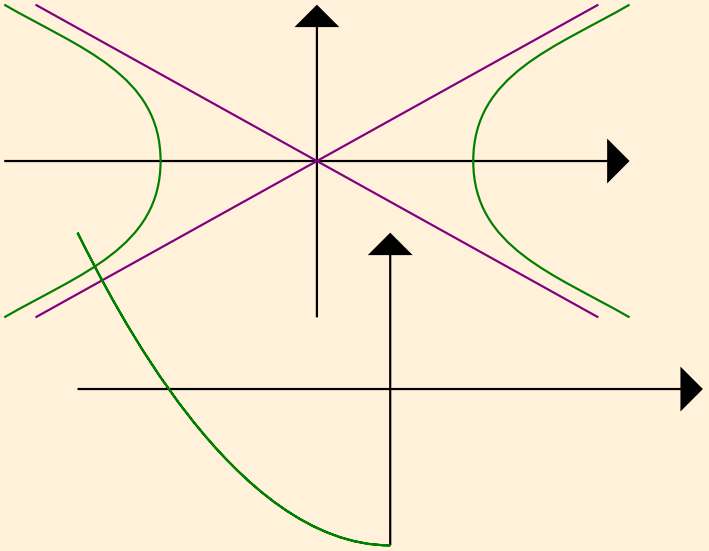
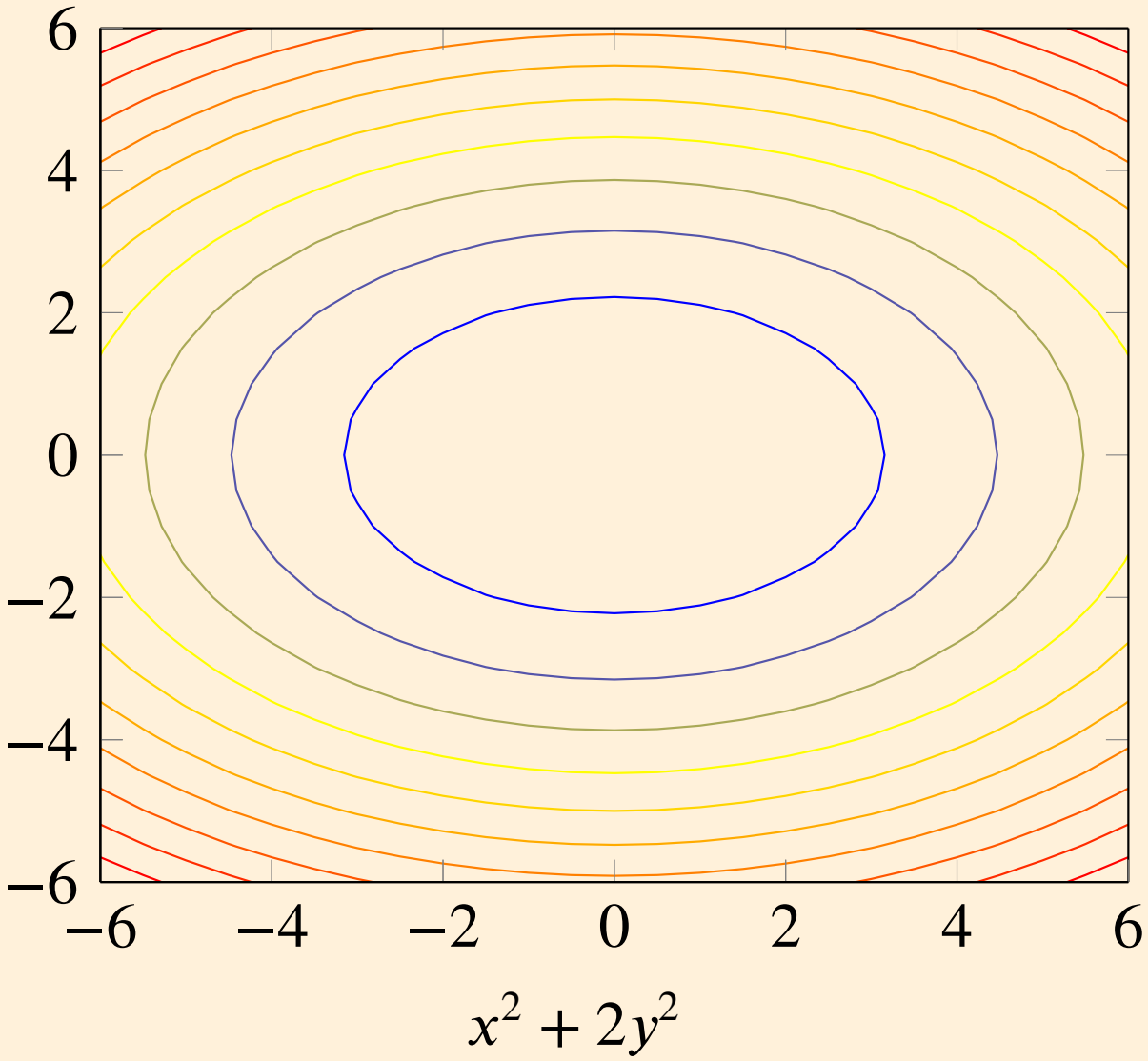
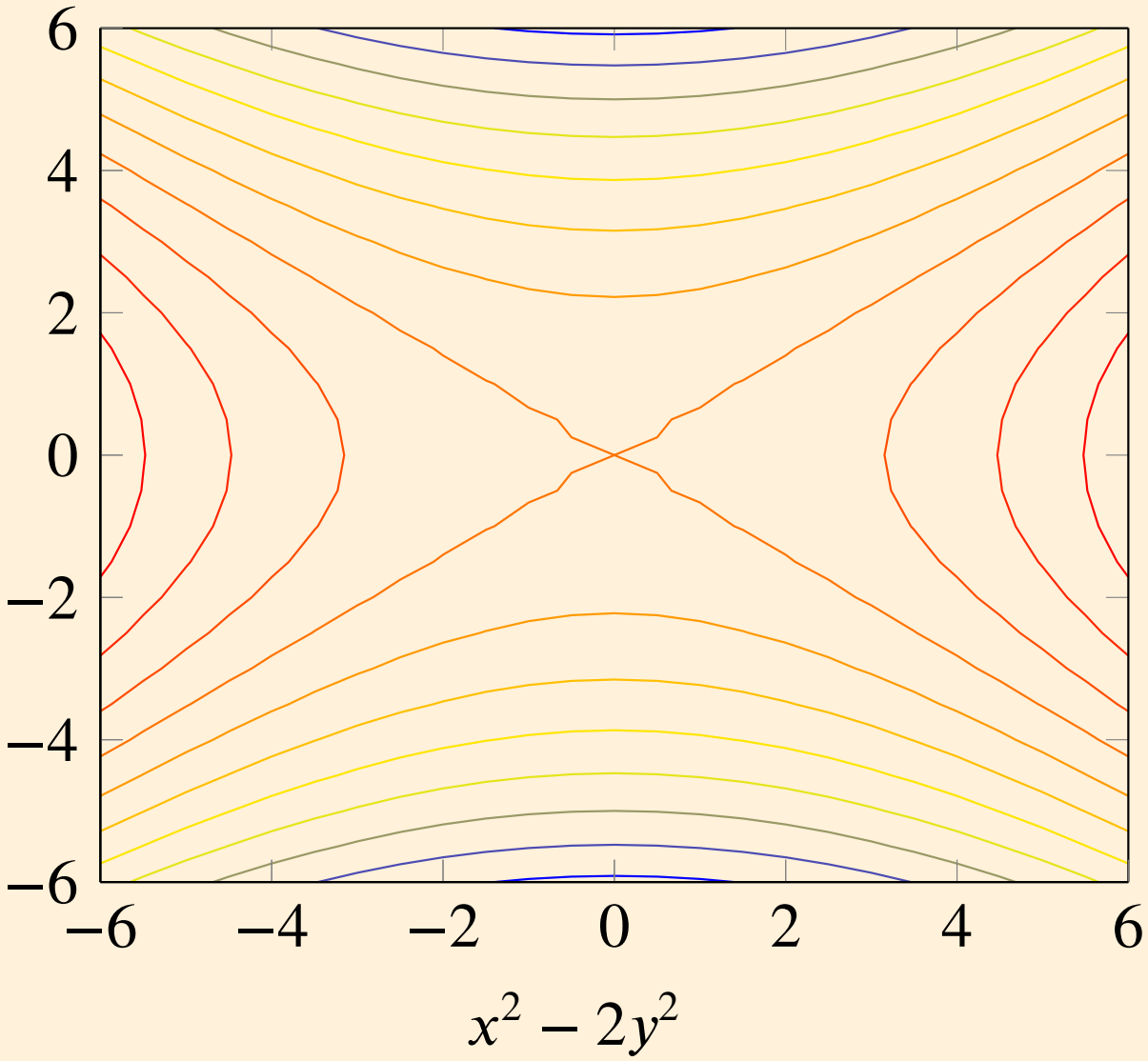


6. $x^2 - 2y = 1$ Parabola

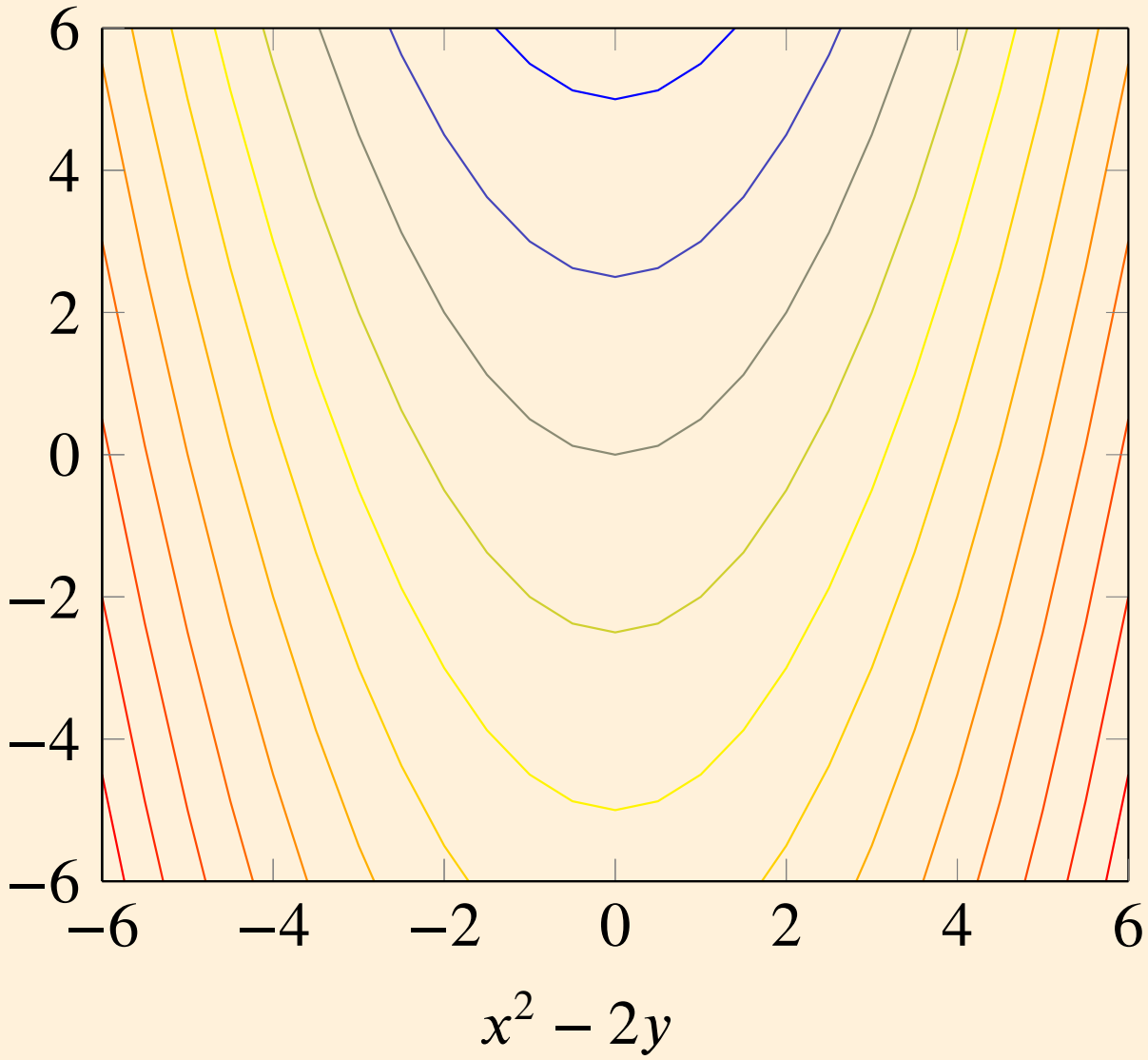
Contour Plots



Contour Plots



Contour Plots



Typical Exam Question: Second Part

- The equation

$$7x^2 + 48xy - 7y^2 - 40x - 30y = 0$$

describes a conic section in the xy–plane. Find a rotated coordinate system (x', y') where the equation of the conic section is in the form

$$\lambda_1(x')^2 + \lambda_2(y')^2 + dx' + ey' = 0.$$

Which type of conic section is it? Draw the new coordinates and the conic section in the xy–plane.

Second Part

$$7x^2 + 48xy - 7y^2 - 40x - 30y = 0$$

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$$\begin{bmatrix} x \\ y \end{bmatrix} \bullet \begin{bmatrix} 7 & 24 \\ 24 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -40 \\ -30 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

Second Part

$$7x^2 + 48xy - 7y^2 - 40x - 30y = 0$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \bullet \begin{bmatrix} 7 & 24 \\ 24 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -40 \\ -30 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = x' \begin{bmatrix} .6 \\ -.8 \end{bmatrix} + y' \begin{bmatrix} .8 \\ .6 \end{bmatrix}$$

Second Part

$$7x^2 + 48xy - 7y^2 - 40x - 30y = 0$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \bullet \begin{bmatrix} 7 & 24 \\ 24 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -40 \\ -30 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$-25x'^2 + 25y'^2 - 40(.6x' + .8y') - 30(-.8x' + .6y') = 0$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = x' \begin{bmatrix} .6 \\ -.8 \end{bmatrix} + y' \begin{bmatrix} .8 \\ .6 \end{bmatrix}$$

Second Part

$$7x^2 + 48xy - 7y^2 - 40x - 30y = 0$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \bullet \begin{bmatrix} 7 & 24 \\ 24 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -40 \\ -30 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$-25x'^2 + 25y'^2 - 40(.6x' + .8y') - 30(-.8x' + .6y') = 0$$

$$-25x'^2 + 25y'^2 - 50y' = 0$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = x' \begin{bmatrix} .6 \\ -.8 \end{bmatrix} + y' \begin{bmatrix} .8 \\ .6 \end{bmatrix}$$

Second Part

$$7x^2 + 48xy - 7y^2 - 40x - 30y = 0$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \bullet \begin{bmatrix} 7 & 24 \\ 24 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -40 \\ -30 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$-25x'^2 + 25y'^2 - 40(.6x' + .8y') - 30(-.8x' + .6y') = 0$$

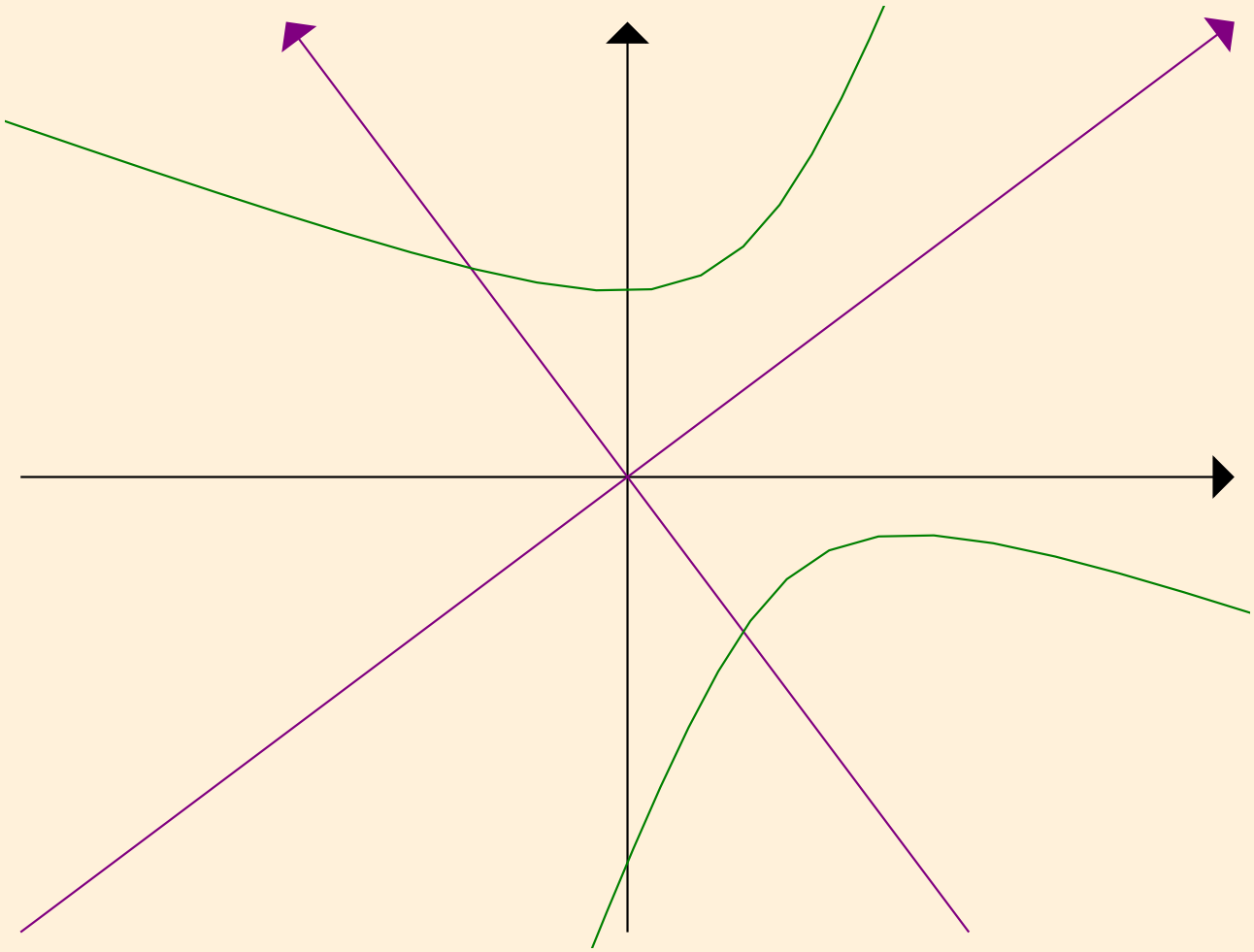
$$-25x'^2 + 25y'^2 - 50y' = 0$$

$$-25x'^2 + 25(y' - 1)^2 = -25$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = x' \begin{bmatrix} .6 \\ -.8 \end{bmatrix} + y' \begin{bmatrix} .8 \\ .6 \end{bmatrix}$$

Second Part

$$-25x'^2 + 25(y' - 1)^2 = -25$$



Summary

- Quadratic forms occur as "level sets" or "trajectories"
- Classification is into: ellipse, hyperbola, parabola
- Classification depends on eigenvalues