Aims and Objectives

By the end of this lecture, you will

▸ know how to classify linear systems by the number of solutions they have

▸ know what it means for a matrix to be invertible

▸ know how to determine invertibility

▸ know how to find the inverse using Gaussian Elimination

Recap

▸ Linear System ↔ Matrix

▸ Use Gaussian Elimination to solve

▸ Combine processes into new processes by:

1. Composition ↔ Matrix multiplication
2. Juxtaposition ↔ Matrix addition

▸ Echelon form:

1. All zero rows are at the bottom
2. The leading term of each non-zero row occurs strictly to the right of the leading term of the row above

▸ Row reduced echelon form:

3. The leading term of each non-zero row is 1
Special Numbers

Recall

If $A$ and $B$ are related by elementary row operations then the corresponding linear systems have exactly the same solutions.

What are the possibilities?

1. All zero rows are at the bottom
   Ignore these in the linear system

2. The leading term of each non-zero row occurs strictly to the right of the leading term of the row above
   Equations stack in order:
   \[
   x_j + 2x_{j+1} + 3x_{j+2} = 4 \\
   x_{j+1} - x_{j+2} = 2
   \]

3. The leading term of each non-zero row is 1
   Equation is of form
   \[
   x_j + 2x_{j+1} + 3x_{j+2} = 4
   \]

Examples

\[
\begin{align*}
2x_1 + 3x_2 - 4x_3 &= -1 \\
x_1 + 4x_2 + 8x_3 &= 3
\end{align*}
\]
\[
\Rightarrow \begin{bmatrix}
2 & 3 & -4 & -1 \\
1 & 4 & 8 & 3
\end{bmatrix}
\]
\[
\Rightarrow \begin{bmatrix}
1 & 4 & 8 & 3 \\
0 & -5 & -20 & -15
\end{bmatrix}
\]
\[
\Rightarrow \begin{bmatrix}
1 & 4 & 8 & 3 \\
0 & 1 & 4 & 3
\end{bmatrix}
\]
\[
x_1 + 4x_2 + 8x_3 = 3 \\
x_1 = -9 + 8t \\
x_2 = 3 - 4t \\
x_3 = t
\]

Number of Solutions

In Echelon Form

Generically: Each equation determines a variable in terms of previously determined variables

But:

1. If a variable hasn’t been determined before it is used, it is “free” and can be set to \textbf{anything}
   
   Infinitely many solutions

2. The last “equation” could be:
   \[
   0x_1 + 0x_2 + \cdots + 0x_n = 1
   \]
   
   which has \textbf{no solutions}
   
   Inconsistent equations

Examples

\[
\begin{align*}
2x_1 + 4x_2 - 4x_3 &= -2 \\
x_1 + 2x_2 - 2x_3 &= 3
\end{align*}
\]
\[
\Rightarrow \begin{bmatrix}
2 & 4 & -4 & -2 \\
1 & 2 & -2 & 3
\end{bmatrix}
\]
\[
\Rightarrow \begin{bmatrix}
1 & 2 & -2 & 3 \\
2 & 4 & -4 & -2
\end{bmatrix}
\]
\[
\Rightarrow \begin{bmatrix}
1 & 2 & -2 & 3 \\
0 & 0 & 0 & -8
\end{bmatrix}
\]
\[
x_1 + 2x_2 - 2x_3 = 3 \\
x_1 + 0x_2 + 0x_3 = -8
\]

No solutions
Examples

\[ 2x_1 + 3x_2 = -1 \]
\[ x_1 + 2x_2 = 3 \]
\[ \Rightarrow \begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & 3 \\ 2 & 3 & -1 \end{bmatrix} \]
\[ \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -7 \end{bmatrix} \]
\[ \Rightarrow x_1 + 2x_2 = 3 \]
\[ 0x_1 - x_2 = -7 \]

Exactly one solution: \( x_1 = -11, x_2 = 7 \)

Partial Information From Partial Knowledge

Question
Can we guess anything about the number of solutions at the start?

In Echelon Form (Coefficient Matrix)

Non-zero row \( \leftrightarrow \) Equation fixing one unique variable

Conclusions

1. #(Non-zero rows) = #(Determined Variables)
2. #(Non-zero rows) ≤ #(Variables) = #(Columns)
3. #(Non-zero rows) ≤ #(Rows)

Size of Matrix and Number of Solutions

\[ A \mathbf{x} = \mathbf{b} \]

\( A \) is an \( m \times n \) matrix

1. If \( n > m \): either no solutions or infinitely many
   ▶ One solution \( \rightarrow \) infinitely many
   ▶ Unique solution \( \rightarrow n \leq m 

2. If \( m > n \): there is a \( \mathbf{b} \) with no solution
   ▶ Always a solution \( \rightarrow m \leq n 

3. Always a unique solution \( \rightarrow m = n \)
Examples and Counterexamples

1. $n > m$:
   1.1 $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow x_1 + 2x_2 + 3x_3 = b_1$
   Lots of solutions
   $x_2 + x_3 = b_2$
   1.2 $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow x_1 + 2x_2 + 3x_3 = b_1$
   No solution or lots of solutions

2. $m < n$:
   $\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \rightarrow x_1 + 2x_2 = b_1$
   $x_2 = b_2 - 2b_1$
   Can always find $b_3$ so that there are no solutions

3. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ No solution with $b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

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Special Case: Exists and is Unique

$Ax = b$ always (any $b$) has a unique solution $\implies m = n$

Definition

A matrix with $(\#(\text{Rows}) = \#(\text{Columns})$ is called square

Partial Converse

For a square matrix $Ax = b$
   always has a solution $\iff$ solutions are unique
   Unique $\iff$ $(\#(\text{Determined Variables}) = \#(\text{All Variables})$ $\iff$ $(\#(\text{Non-zero rows in rref}) = \#(\text{Columns})$ $\iff$ $(\#(\text{Non-zero rows in rref}) = \#(\text{Rows})$ $\iff$ No equations having no variables $(0 = b_n)$ $\iff$ Always has a solution

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Back to the Factories

Factory to Depot

Passing Through the Depots

$\begin{bmatrix} F1 & F2 \\ D1 & .29 & .35 \\ D2 & .71 & .65 \end{bmatrix}$

Recall: matrix records proportions

Question

Can we find out the productivity of the factories by measuring what arrives at the depots?

Solve:
   
   \[
   .29p_1 + .35p_2 = d_1 \\
   .71p_1 + .65p_2 = d_2
   \]

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Solution

\[
\begin{bmatrix}
.29 & .35 \\
.71 & .65
\end{bmatrix}
\xrightarrow{R_2 \rightarrow .29R_2}
\begin{bmatrix}
.29 & .35 \\
.2059 & 1.885
\end{bmatrix}
\xrightarrow{R_1 \rightarrow .71R_1}
\begin{bmatrix}
.29 & .35 \\
0 & -0.66
\end{bmatrix}
\xrightarrow{R_1 \rightarrow -.06R_1}
\begin{bmatrix}
-0.0174 & -0.0210 \\
0 & 0.66
\end{bmatrix}
\xrightarrow{R_1 \rightarrow -.35R_2}
\begin{bmatrix}
-0.0174 & 0 \\
0 & -0.06
\end{bmatrix}
\xrightarrow{R_1 \rightarrow .29^{-1} R_1}
\begin{bmatrix}
-0.66 & 0.66 \\
0 & 0.66
\end{bmatrix}
\]

Back and Forth

\[.29p_1 + .35p_2 = d_1 \quad \rightarrow \quad \frac{-100}{6} (.65d_1 - .35d_2) = p_1\]
\[.71p_1 + .65p_2 = d_2 \quad \rightarrow \quad \frac{-100}{6} (-.35d_1 + .29d_2) = p_2\]

Remark

1. Can figure out what leaves factories from what arrives at depots.
2. Can arrange for any desired arrival amounts by adjusting productions.
3. No presumption of cause or effect.

Invertible

Definition

A process is said to be invertible if

1. each input is uniquely determined by its output, and
2. each potential output is possible.

A matrix is invertible if it represents an invertible process

1. Each input is uniquely determined by its output
   \(Ax = b\) has at most one solution, so \(m \geq n\)
2. Each potential output is possible
   \(Ax = b\) has at least one solution, so \(m \leq n\)

Invertible matrix \(\leftrightarrow\) square

Invertible Examples

1. \[
\begin{bmatrix}
.29 & .35 \\
.71 & .65
\end{bmatrix}
\]
   represents an invertible process

2. \[
\begin{bmatrix}
.29 & .35 \\
.58 & .70
\end{bmatrix}
\]
   is not invertible: \[
\begin{bmatrix}
-.35 \\
-.29
\end{bmatrix}
\] and \[
\begin{bmatrix}
-.70 \\
-.58
\end{bmatrix}
\] both go to \(\begin{bmatrix}0 \\ 0\end{bmatrix}\)

3. \[
\begin{bmatrix}
.29 & .35 \\
.58 & .70
\end{bmatrix}
\]
   is not invertible: no way to get \(\begin{bmatrix}1 \\ 0\end{bmatrix}\)
An Incredibly Important Process

Introducing

The “do nothing” process!

Output = Input

\[ y_1 = x_1, y_2 = x_2, \ldots, y_n = x_n \]

Representing matrix:

\[
I_n \coloneqq \begin{bmatrix}
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{bmatrix}
\]

(Technically, one for each size \( n \))

Inverses

Rough Definition

An inverse of a process is another process such that composing either way around results in the “do nothing” process.

In matrix language: an inverse of \( A \) is \( B \) such that \( AB = I_m \) and \( BA = I_n \).

Lemma

A matrix is invertible if and only if it has an inverse.

Computing Inverses

Does

\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 4 \\
3 & 4 & 6
\end{bmatrix}
\]

have an inverse?

If so, \( Ax = b \) has a solution for any \( b \).

For example … \[
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\]

or

\[
\begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}
\]

Let’s see if it does.

Simultaneous Gaussian Elimination

\[
\begin{bmatrix}
1 & 2 & 3 & 1 & 0 & 0 \\
2 & 3 & 4 & 0 & 1 & 0 \\
3 & 4 & 6 & 0 & 0 & 1
\end{bmatrix}
\]

\[
R_2 \rightarrow -2R_1 \\
R_3 \rightarrow -3R_1
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & -3 & 2 \\
0 & 0 & 1 & 1 & -2
\end{bmatrix}
\]

\[
R_3 \rightarrow R_2 + R_3 \\
R_2 \rightarrow R_2 + R_1
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 3 \\
0 & 0 & 1 & 1 & -2
\end{bmatrix}
\]
Check Your Answer

Is

\[
B = \begin{bmatrix}
-2 & 0 & 1 \\
0 & 3 & -2 \\
1 & -2 & 1
\end{bmatrix}
\]

an inverse for \( A \)?

Check: \( AB = I_3 \) and \( BA = I_3 \)

Alternative argument: write \( B = [b_1, b_2, b_3] \) then \( Ab_i = e_i \) so \( AB = [e_1, e_2, e_3] = I_3 \)

Summary

- Matrix manipulations follow from what happens to processes
- Invertible matrices correspond to invertible processes
- Invertible process means input and output determine each other