TMA4115 Matematikk 3

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Spring 2011

Lecture 26: Quadratic Forms

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5th April 2011

Aims and Objectives

By the end of this lecture, you will

- have seen the classification of quadratic forms
- have seen how to relate them to conic sections

Recap: Symmetric Matrices

Definition

$$\text{Symmetric: } A^\top = A$$

Theorem

$$\text{Symmetric} \iff \text{orthogonal basis of eigenvectors}$$

$$f(x+h) \simeq f(x) + \nabla f \bullet h + h \bullet Hfh$$

Typical Exam Question

Question

- Let $A = \begin{bmatrix} 7 & 24 \\ 24 & -7 \end{bmatrix}$. Find the eigenvalues of A and eigenvectors v_1 and v_2 such that the matrix P with column vectors v_1 and v_2 is an orthogonal matrix with determinant 1.
- The equation $7x^2 + 48xy - 7y^2 - 40x - 30y = 0$ describes a conic section in the xy–plane. Find a rotated coordinate system (x', y') where the equation of the conic section is in the form $\lambda_1(x')^2 + \lambda_2(y')^2 + dx' + ey' = 0$. Which type of conic section is it? Draw the new coordinates and the conic section in the xy–plane.

First Part

$$A = \begin{bmatrix} 7 & 24 \\ 24 & -7 \end{bmatrix}$$

$$c(\lambda) = \lambda^2 - (7-7)\lambda + (-49 - 24^2) = \lambda^2 - 625 = (\lambda - 25)(\lambda + 25)$$

Eigenvalues: $25, -25$
Eigenvectors:

$$\begin{bmatrix} -18 & 24 \\ 24 & -32 \end{bmatrix} \to \begin{bmatrix} -3 & 4 \\ 3 & -4 \end{bmatrix} \to \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

Other **must** be $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$.

First Part (continued)

such that the matrix P with column vectors v_1 and v_2 is an **orthogonal** matrix with determinant 1.

Orthogonal matrix means columns are ortho**normal**

$$\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} \to \begin{bmatrix} -.6 & .8 \\ .8 & .6 \end{bmatrix} \to \begin{bmatrix} .6 & .8 \\ -.8 & .6 \end{bmatrix} \text{ or } \begin{bmatrix} .8 & -.6 \\ .6 & .8 \end{bmatrix}$$

1. Columns **orthogonal** not **orthonormal**
2. Determinant $-.6 \times .6 - .8 \times .8 = -.36 - .64 = -1$
3. Not unique

Second Part

- The equation

$$7x^2 + 48xy - 7y^2 - 40x - 30y = 0$$

describes a conic section in the xy–plane. Find a rotated coordinate system (x', y') where the equation of the conic section is in the form

$$\lambda_1(x')^2 + \lambda_2(y')^2 + dx' + ey' = 0.$$

Which type of conic section is it? Draw the new coordinates and the conic section in the xy–plane.

Solving an ODE

Planetary Motion:

$$F = ma = -\frac{GMm}{r^2}$$

In vector form:

$$u'' = -\frac{1}{\|u\|^3} u$$

Convert to first order:

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} v \\ -\frac{1}{\|u\|^3} u \end{bmatrix}$$

(Note that this is now **six** dimensional!)

Constants of Motion

Observation: Planets stay on the ecliptic plane.
How do we encode that mathematically? A plane is determined by its normal vector.

$$\begin{aligned} w &= u \times v \\ w' &= u \times v' + u' \times v \\ &= u \times \left(-\frac{1}{\|u\|^3} u \right) + v \times v \\ &= 0 \end{aligned}$$

Conclusion: Initial conditions $u(0)$ and $u'(0)$ define a plane (generically) and the motion stays in that plane.

Rotate space so that this is the xy–plane.

Six dimensions $\longrightarrow$ **four**

Constants of Motion

Observation: Energy is conserved.
How do we encode that mathematically?

$$E = \frac{1}{2}\|v\|^2 - \frac{1}{\|u\|}$$

$$\begin{aligned} E &= \frac{1}{2} v \bullet v - (u \bullet u)^{-1/2} \\ E' &= v \bullet v' + 1/2 (u \bullet u)^{-3/2} (u' \bullet u + u \bullet u') \\ &= u' \bullet \left(-\frac{1}{\|u\|^3} u \right) + u' \bullet \left(\frac{1}{\|u\|^3} u \right) \\ &= 0 \end{aligned}$$

Conservation of Energy

$$\frac{1}{2}\|v\|^2 - \frac{1}{\|u\|} = E = \text{constant}$$

Question: How does that help?

$$\begin{aligned} E &= \frac{1}{2}\|v\|^2 - \frac{1}{\|u\|} & &= \frac{1}{2} v \bullet v - \frac{1}{\|u\|^3} u \bullet u \\ &= \frac{1}{2} u' \bullet v + v' \bullet u & &= \begin{bmatrix} u' \\ v' \end{bmatrix} \bullet \begin{bmatrix} \frac{1}{2} v \\ u \end{bmatrix} \\ &= \begin{bmatrix} u' \\ v' \end{bmatrix} \bullet \begin{bmatrix} 0 & \frac{1}{2} I \\ I & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} & &= \frac{1}{2} \left(\begin{bmatrix} u \\ v \end{bmatrix} \bullet \begin{bmatrix} 0 & \frac{1}{2} I \\ I & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \right)' \end{aligned}$$

Conservation of Energy

Conclusion:

$$\begin{bmatrix} u \\ v \end{bmatrix} \bullet \begin{bmatrix} 0 & \frac{1}{2}I \\ I & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + c \bullet \begin{bmatrix} u \\ v \end{bmatrix} = \text{const}$$

Not **quite** symmetric, but almost.
Basis of Eigenvectors:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ \sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -\sqrt{2} \end{bmatrix} \right\}$$

Eigenvalues: $\frac{1}{\sqrt{2}}$,$-\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{2}}$, $-\frac{1}{\sqrt{2}}$

Quadratic Forms from ODEs

More Generally: ODE $u' = G(u)$ can be difficult to solve so look for **conservation laws**.

Suppose $G(u) \perp Au$ for some symmetric matrix A

$$\Big(\text{ diagonalisable would do } \Big)$$

Then $(u \bullet Au)' = 2u' \bullet Au = 2G(u) \bullet Au = 0$

So $u \bullet Au$ is **constant** and the solution of $u' = G(u)$ lies on the level set of $u \bullet Au$.

Quadratic Forms

Definition
General Quadratic Form (2D):

$$ax^2 + 2bxy + cy^2 + dx + ey + f = g$$

Matrix Reformulation:

$$\begin{bmatrix} x \\ y \end{bmatrix} \bullet \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} d \\ e \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = g$$

Classification

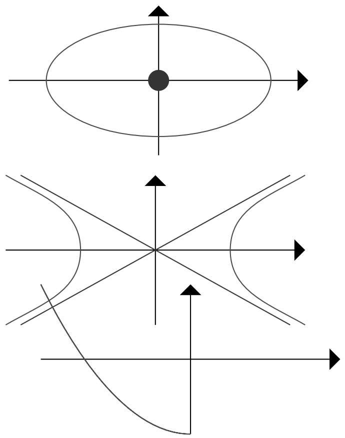
Procedure:

1. Find orthogonal eigenbasis of $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$
2. Write $\begin{bmatrix} x \\ y \end{bmatrix} = Xv_1 + Yv_2$
3. $\begin{bmatrix} x \\ y \end{bmatrix} \bullet \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda_1 X^2 + \lambda_2 Y^2$
4. $\begin{bmatrix} d \\ e \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = DX + EY$
5. $\lambda_1 X^2 + \lambda_2 Y^2 + DX + EY = \lambda_1 (X - D/2\lambda_1)^2 + \lambda_2 (Y - E/2\lambda_2)^2 - G$
6. Translate: $\overline{X} = X - D/2\lambda_1$, $\overline{Y} = Y - E/2\lambda_2$
7. Original system is equivalent to $\lambda_1 \overline{X}^2 + \lambda_2 \overline{Y}^2 = G$, but rotated and translated

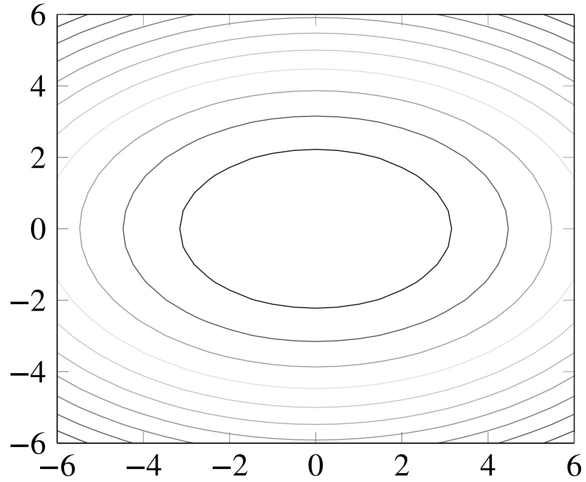
Classification

Possibilities:

1. $x^2 + 2y^2 = 1$ Ellipse
2. $x^2 + 2y^2 = 0$ Point
3. $x^2 + 2y^2 = -1$ No Solutions
4. $x^2 - 2y^2 = 1$ Hyperbola
5. $x^2 - 2y^2 = 0$ Lines
6. $x^2 - 2y = 1$ Parabola

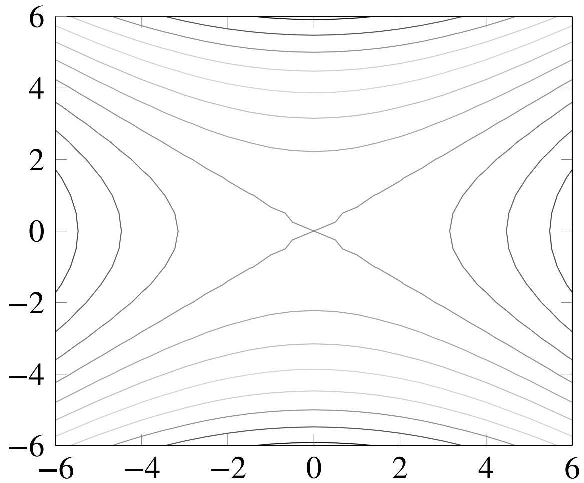


Contour Plots



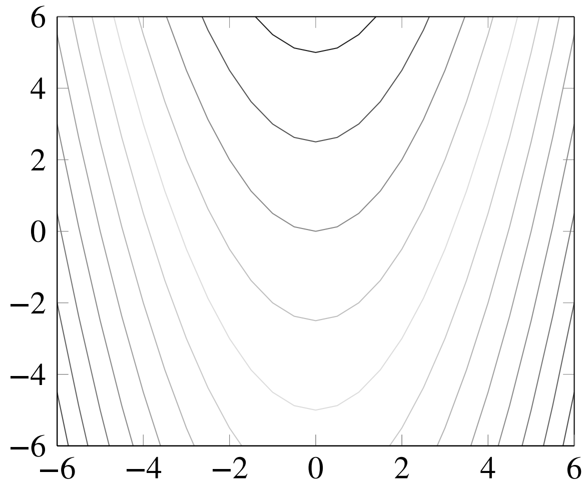
$x^2 + 2y^2$

Contour Plots



$x^2 - 2y^2$

Contour Plots



$x^2 - 2y$

Typical Exam Question: Second Part

- The equation

$$7x^2 + 48xy - 7y^2 - 40x - 30y = 0$$

describes a conic section in the xy–plane. Find a rotated coordinate system (x', y') where the equation of the conic section is in the form

$$\lambda_1(x')^2 + \lambda_2(y')^2 + dx' + ey' = 0.$$

Which type of conic section is it? Draw the new coordinates and the conic section in the xy–plane.

Second Part

$$7x^2 + 48xy - 7y^2 - 40x - 30y = 0$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \bullet \begin{bmatrix} 7 & 24 \\ 24 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -40 \\ -30 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$-25x'^2 + 25y'^2 - 40(.6x' + .8y') - 30(-.8x' + .6y') = 0$$

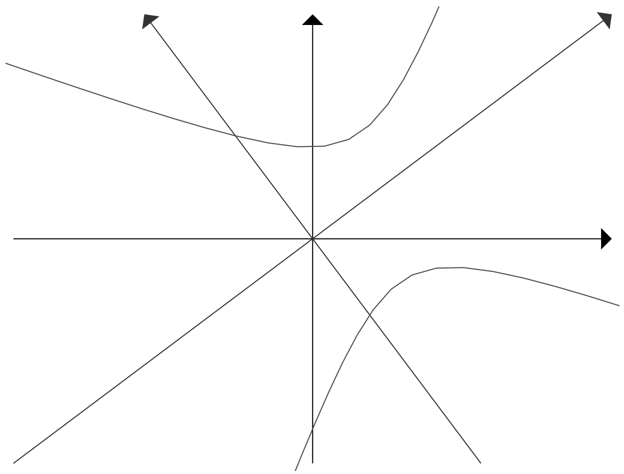
$$-25x'^2 + 25y'^2 - 50y' = 0$$

$$-25x'^2 + 25(y' - 1)^2 = -25$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = x' \begin{bmatrix} .6 \\ -.8 \end{bmatrix} + y' \begin{bmatrix} .8 \\ .6 \end{bmatrix}$$

Second Part

$$-25x'^2 + 25(y' - 1)^2 = -25$$



Summary

- Quadratic forms occur as “level sets” or “trajectories”
- Classification is into: ellipse, hyperbola, parabola
- Classification depends on eigenvalues