Aims and Objectives

By the end of this lecture, you will
▶ have seen the classification of quadratic forms
▶ have seen how to relate them to conic sections

Recap: Symmetric Matrices

Definition
Symmetric: \( A^T = A \)

Theorem
Symmetric \( \iff \) orthogonal basis of eigenvectors

\[
f(x + h) \approx f(x) + \nabla f \cdot h + \frac{1}{2} h \cdot H f h
\]
Typical Exam Question

Question

▶ Let \( A = \begin{bmatrix} 7 & 24 \\ 24 & -7 \end{bmatrix} \). Find the eigenvalues of \( A \) and eigenvectors \( v_1 \) and \( v_2 \) such that the matrix \( P \) with column vectors \( v_1 \) and \( v_2 \) is an orthogonal matrix with determinant 1.
▶ The equation \( 7x^2 + 48xy - 7y^2 - 40x - 30y = 0 \) describes a conic section in the \( xy \)-plane. Find a rotated coordinate system \( (x', y') \) where the equation of the conic section is in the form \( \lambda_1(x')^2 + \lambda_2(y')^2 + dx' + ey' = 0 \). Which type of conic section is it? Draw the new coordinates and the conic section in the \( xy \)-plane.

First Part

\[
A = \begin{bmatrix} 7 & 24 \\ 24 & -7 \end{bmatrix}
\]

\[
c(\lambda) = \lambda^2 - (7 - 7)\lambda + (-49 - 24^2) = \lambda^2 - 625 = (\lambda - 25)(\lambda + 25)
\]

Eigenvalues: 25, -25

Eigenvectors:

\[
\begin{bmatrix} -18 & 24 \\ 24 & -32 \end{bmatrix} \rightarrow \begin{bmatrix} -3 & 4 \\ 3 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 4 \\ 3 \end{bmatrix}
\]

Other must be \( \begin{bmatrix} -3 \\ 4 \end{bmatrix} \).

First Part (continued)

such that the matrix \( P \) with column vectors \( v_1 \) and \( v_2 \) is an orthogonal matrix with determinant 1.

Orthogonal matrix means columns are orthonormal

\[
\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} -.6 & .8 \\ .8 & .6 \end{bmatrix} \rightarrow \begin{bmatrix} .6 & .8 \\ -.8 & .6 \end{bmatrix} \text{ or } \begin{bmatrix} .8 & -.6 \\ .6 & .8 \end{bmatrix}
\]

1. Columns orthogonal not orthonormal
2. Determinant \(-.6 \times .6 - .8 \times .8 = -.36 - .64 = -1\)
3. Not unique

Second Part

▶ The equation

\[
7x^2 + 48xy - 7y^2 - 40x - 30y = 0
\]

describes a conic section in the \( xy \)-plane. Find a rotated coordinate system \( (x', y') \) where the equation of the conic section is in the form

\[
\lambda_1(x')^2 + \lambda_2(y')^2 + dx' + ey' = 0.
\]

Which type of conic section is it? Draw the new coordinates and the conic section in the \( xy \)-plane.
Solving an ODE

Planetary Motion:

\[ F = ma = -\frac{GMm}{r^2} \]

In vector form:

\[ u'' = -\frac{1}{\|u\|^3} u \]

Convert to first order:

\[
\begin{bmatrix}
  u' \\
  v'
\end{bmatrix} = \begin{bmatrix}
  -\frac{1}{\|u\|^3} v \\
  -\frac{1}{\|u\|^3} u
\end{bmatrix}
\]

(Note that this is now six dimensional!)

Constants of Motion

Observation: Planets stay on the ecliptic plane. How do we encode that mathematically? A plane is determined by its normal vector.

\[ w = u \times v \]

\[ w' = u \times v' + u' \times v \]

\[ = u \times \left( -\frac{1}{\|u\|^3} u \right) + v \times v \]

\[ = 0 \]

Conclusion: Initial conditions \( u(0) \) and \( u'(0) \) define a plane (generically) and the motion stays in that plane.

Rotate space so that this is the \( xy \)-plane.

Six dimensions \( \rightarrow \) four

Conservation of Energy

Observation: Energy is conserved. How do we encode that mathematically?

\[ E = \frac{1}{2} \|v\|^2 - \frac{1}{\|u\|} \]

\[ E = \frac{1}{2} v \cdot v - (u \cdot u)^{-\frac{1}{2}} \]

\[ E' = v \cdot v' + \frac{1}{2} (u \cdot u)^{-\frac{1}{2}} (u' \cdot u + u \cdot u') \]

\[ = u' \cdot \left( -\frac{1}{\|u\|^3} u \right) + u' \cdot \left( \frac{1}{\|u\|^3} u \right) \]

\[ = 0 \]
Conservation of Energy

Conclusion:
\[
\begin{bmatrix}
u \\
v
\end{bmatrix} \cdot \begin{bmatrix}
0 & \frac{1}{2} f \\
f & 0
\end{bmatrix} \begin{bmatrix}
u \\
v
\end{bmatrix} + c \cdot \begin{bmatrix}
u \\
v
\end{bmatrix} = \text{const}
\]

Not quite symmetric, but almost.

Basis of Eigenvectors:
\[
\left\{ \begin{bmatrix}
1 \\
0 \\
\frac{1}{\sqrt{2}} \\
0
\end{bmatrix},
\begin{bmatrix}
1 \\
0 \\
-\frac{1}{\sqrt{2}} \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
1 \\
0 \\
\sqrt{2}
\end{bmatrix},
\begin{bmatrix}
0 \\
1 \\
0 \\
-\sqrt{2}
\end{bmatrix}\right\}
\]

Eigenvalues:
\[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\]

Quadratic Forms from ODEs

More Generally: ODE \(u' = G(u)\) can be difficult to solve so look for conservation laws.

Suppose \(G(u) \perp Au\) for some symmetric matrix \(A\)

\[
\left( \text{diagonalisable would do} \right)
\]

Then \((u \cdot Au)' = 2u' \cdot Au = 2G(u) \cdot Au = 0\)

So \(u \cdot Au\) is constant and the solution of \(u' = G(u)\) lies on the level set of \(u \cdot Au\).

Quadratic Forms

Definition
General Quadratic Form (2D):
\[ax^2 + 2bxy + cy^2 + dx + ey + f = g\]

Matrix Reformulation:
\[
\begin{bmatrix}
x \\
y
\end{bmatrix} \cdot \begin{bmatrix}
a & b \\
b & c
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix} + \begin{bmatrix}
d \\
e
\end{bmatrix} \cdot \begin{bmatrix}
x \\
y
\end{bmatrix} = g
\]

Classification

Procedure:
1. Find orthogonal eigenbasis of \(\begin{bmatrix} a & b \\ b & c \end{bmatrix}\)
2. Write \(\begin{bmatrix} x \\ y \end{bmatrix} = Xv_1 + Yv_2\)
3. \(\begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda_1 X^2 + \lambda_2 Y^2\)
4. \(\begin{bmatrix} d \\ e \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = DX + EY\)
5. \(\lambda_1 X^2 + \lambda_2 Y^2 + DX + EY = \lambda_1 (X - \eta_2 \lambda_1)^2 + \lambda_2 (Y - \eta_2 \lambda_2)^2 - G\)
6. Translate: \(\overline{X} = X - \eta_2 \lambda_1, \overline{Y} = Y - \eta_2 \lambda_2\)
7. Original system is equivalent to \(\lambda_1 \overline{X}^2 + \lambda_2 \overline{Y}^2 = G\), but rotated and translated
Classification

Possibilities:
1. $x^2 + 2y^2 = 1$ Ellipse
2. $x^2 + 2y^2 = 0$ Point
3. $x^2 + 2y^2 = -1$ No Solutions
4. $x^2 - 2y^2 = 1$ Hyperbola
5. $x^2 - 2y^2 = 0$ Lines
6. $x^2 - 2y = 1$ Parabola

Contour Plots
Typical Exam Question: Second Part

- The equation
  
  \[ 7x^2 + 48xy - 7y^2 - 40x - 30y = 0 \]

  describes a conic section in the \( xy \)-plane. Find a rotated coordinate system \((x', y')\) where the equation of the conic section is in the form

  \[ \lambda_1(x')^2 + \lambda_2(y')^2 + dx' + ey' = 0. \]

  Which type of conic section is it? Draw the new coordinates and the conic section in the \( xy \)-plane.

Second Part

- The equation
  
  \[ 7x^2 + 48xy - 7y^2 - 40x - 30y = 0 \]

  can be written as

  \[
  \begin{bmatrix}
  x \\
  y
  \end{bmatrix} \cdot \begin{bmatrix}
  7 & 24 \\
  24 & -7
  \end{bmatrix} \begin{bmatrix}
  x \\
  y
  \end{bmatrix} + \begin{bmatrix}
  -40 \\
  -30
  \end{bmatrix} \cdot \begin{bmatrix}
  x \\
  y
  \end{bmatrix} = 0
  \]

  \[ -25x'^2 + 25y'^2 - 40(0.6x' + 0.8y') - 30(-0.8x' + 0.6y') = 0 \]

  \[ -25x'^2 + 25y'^2 - 50y' = 0 \]

  \[ -25x'^2 + 25(y' - 1)^2 = -25 \]

  \[
  \begin{bmatrix}
  x \\
  y
  \end{bmatrix} = x' \begin{bmatrix}
  0.6 \\
  -0.8
  \end{bmatrix} + y' \begin{bmatrix}
  0.8 \\
  0.6
  \end{bmatrix}
  \]

Second Part

- \[-25x'^2 + 25(y' - 1)^2 = -25\]

Summary

- Quadratic forms occur as “level sets” or “trajectories”
- Classification is into: ellipse, hyperbola, parabola
- Classification depends on eigenvalues