TMA4115 Matematikk 3

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Lecture 1: Complex Numbers

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Aims and Objectives

By the end of this lecture, you will

▶ have seen the definition of a complex number
▶ have seen the rules for computing with complex numbers
▶ have seen examples of using those rules
Boring Admin

Key Facts

- All information on website
- Tuesday lectures start at 8:30am
- Forum for questions and comments
- Reference group
Complex Numbers: Typical Exam Question

Question
Write the complex number \( w = \frac{3-\text{i}}{2\text{i}-1} \) in polar form. Find all the solutions of the equation \( z^4 = w \) and draw the solutions in the complex plane.

Next Question

1. What is a complex number?
2. What is \( \text{i} \)?
3. What is polar form?
4. enough for now!
What is a Complex Number?

Working Definition

Something with $\sqrt{-1}$ in it.

Problem: Already ambiguous.

What is $\sqrt{1}$?

Solution: introduce $i$ such that $i^2 = -1$

$\sqrt{-1}$ is a property, $i$ is a thing with that property
Why Bother with Complex Numbers?

Mathematics is a language:

- Words ↔ Definitions
- Sentences ↔ Theorems

(Roughly)

Need words to express ideas
Words have no intrinsic reality;
their worth comes from their usefulness.

- Does “dog” exist? Separates “dog food” from “cat food”
- Does “$\sqrt{-1}$” exist?

Wrong Question!
Is $\sqrt{-1}$ Useful?

Real Question
What concepts does $\sqrt{-1}$ allow us to talk about?

- Roots of any polynomial, e.g. $x^2 + 1$
  Seems boring, but incredibly useful
- Signal analysis
- Electromagnetism
- Quantum Theory
- ...

*The shortest path between two truths in the real domain passes through the complex plane.*

Hadamard
Is $\sqrt{-1}$ Useful?

**Question**

How do we make $\sqrt{-1}$ useful?

**Important Point:** $i$ by itself is not useful, it only becomes useful when we know that $i^2 = -1$.

**Form Versus Function**

**Slogan:** Mathematical objects are determined more by what they do than what they are.

**Question**

What can we do with $i$? Apart from squaring it?
Extending Real Numbers

Purpose of the Starship Enterprise
Space: the final frontier. These are the voyages of the starship Enterprise. Its five-year mission: to explore strange new worlds, to seek out new life and new civilizations, to boldly go where no man has gone before.

Key Point
Complex numbers extend real numbers

Question
So what can we do with real numbers?
Things to do with Real Numbers

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Things to do with Complex Numbers

What we know so far:

1. \( \mathbb{C} \) extends \( \mathbb{R} \) \( \implies \mathbb{R} \subseteq \mathbb{C} \)
2. \( i \in \mathbb{C} \)
3. Want to be able to:
   - Add
   - Subtract
   - Multiply
   - Divide (if non-zero)

Proto–Definition

\( \mathbb{C} \) consists of \( \mathbb{R} \) and \( i \) together with everything needed for the operations.
Addition

$\mathbb{R} \subseteq \mathbb{C}, \ i \in \mathbb{C} \implies 3 + i \in \mathbb{C}$

More generally: for $x \in \mathbb{R}$, must have $x + i \in \mathbb{C}$

Not enough: also need $i + i$ and $i + i + i$ ...
(Throw in subtraction)

Conclusion: for $x \in \mathbb{R}$ and $k \in \mathbb{Z}$, must have

$x + ki \in \mathbb{C}$
Multiplication

\[ \mathbb{R} \subseteq \mathbb{C}, \ i \in \mathbb{C} \implies 3i \in \mathbb{C} \]

More generally: for \( y \in \mathbb{R} \), must have \( yi \in \mathbb{C} \)

Not enough: already have stuff like \( \pi + 3i \):

\[ \pi + 3i \in \mathbb{C}, \ i \in \mathbb{C} \implies -3 + \pi i \in \mathbb{C} \]

\[ \pi e + i \in \mathbb{C}, \ e \in \mathbb{C} \implies \pi + ei \in \mathbb{C} \]

Conclusion: for \( x, y \in \mathbb{R} \) and \( k \in \mathbb{Z} \), must have

\[ x + ki \in \mathbb{C} \]
Is It Enough?

▶ Addition:

\[(x + iy) + (u + iv) = (x + u) + i(y + v)\]

▶ Subtraction:

\[(x + iy) - (u + iv) = (x - u) + i(y - v)\]

▶ Multiplication:

\[(x + iy) \times (u + iv) = xu + iyu + xiv + iyi
\[= (xu - yv) + i(yu + xv)\]

▶ Division:  ?
Division

Enough to do reciprocals:

\[
\frac{x + iy}{u + iv} = (x + iy) \frac{1}{u + iv}
\]

Case Study

\[
\frac{1}{3 + i2} = ?
\]

\[
1 = (3 + i2)(u + iv)
\]

\[
1 + i0 = (3u - 2v) + i(2u + 3v)
\]

\[
\begin{aligned}
1 &= 3u - 2v \\
0 &= 2u + 3v
\end{aligned}
\]

\[
\implies u = \frac{3}{13}, \quad v = -\frac{2}{13}
\]
Case Study

\[
\frac{1}{3 + 2i} = u + iv
\]

\[
1 = (3 + 2i)(u + iv)
\]

\[
1 + i0 = (3u - 2v) + i(2u + 3v)
\]

\[
\begin{align*}
1 &= 3u - 2v \\
0 &= 2u + 3v
\end{align*}
\]

\[
\Rightarrow \quad u = \frac{3}{13}, \quad v = \frac{-2}{13}
\]

\[
\frac{1}{x + iy} = \frac{x}{x^2 + y^2} + \frac{-y}{x^2 + y^2} = \frac{1}{x^2 + y^2}(x - iy)
\]
What Is a Complex Number?

- What we get from $\mathbb{R}$ and $i$ by adding and multiplying
- Anything of the form $x + iy$
- An element of the algebraically closed extension of $\mathbb{R}$

Who cares what it is? Only care about what it does.

What it looks like
Whatever a complex number is, it can be represented uniquely as a pair of real numbers which we write in the form $x + iy$. 
The Complex Plane

\[ z \leftrightarrow x + iy \leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} \leftrightarrow (r, \theta) \]

Known as the Complex Plane
### Polar Coordinates

\[ z \in \mathbb{C} \leftrightarrow x + iy \leftrightarrow r(\cos(\theta) + i\sin(\theta)) \]

Different **representations** of a complex number

<table>
<thead>
<tr>
<th>( z )</th>
<th>Cartesian</th>
<th>Polar</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x + iy )</td>
<td>(( r, \theta ))</td>
<td>horrendous</td>
</tr>
</tbody>
</table>

- **Add:** \( z + w \)
  \[(x + u) + i(y + v)\]
- **Multiply:** \( zw \)
  \[(xu - yv) + i(xv + yu)\]
- **Reciprocal:** \( z^{-1} \)
  \[\frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2}\]

Better notation to come
What are the Complex Numbers?

Complex Numbers: Intrinsic View
Pairs of real numbers with \( \begin{bmatrix} 0 \\ 1 \end{bmatrix}^2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \).

Complex Numbers: Extrinsic View
Pairs of real numbers with a rotation \( J : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} -y \\ x \end{bmatrix} \)

So complex numbers are likely to play a role anywhere that there is a fixed rotation.
Summary

Complex Numbers ...

- are as real as real numbers
- can be represented in the form $x + iy$ with $x, y \in \mathbb{R}$
- have (almost) the same basic operations as $\mathbb{R}$, subject to the same rules
- can be represented in polar coordinates

Next Time ...

- powers, and
- exponentials