Aims and Objectives

By the end of this lecture, you will

▸ have seen how powers work with complex numbers

▸ have seen a definition of the complex exponential function
Recap: Complex Numbers

Complex Numbers

\[\mathbb{C} \ni i \cdot i^2 = -1\]

\[z \leftrightarrow x + iy\]
Recap: Addition, Multiplication, and Division

- Addition: $z + w = (x + u) + i(y + v)$
- Multiplication: $zw = (xu - yv) + i(xv + yu)$
- Reciprocal: $\frac{1}{z} = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2}$
Recap: Polar Coordinates and the Complex Plane

\[ z \leftrightarrow x + iy \leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} \leftrightarrow (r, \theta) \]
Taking Powers

The Next Step

1. Addition: \( 2 + 3 = 5 \)

2. Repeated Addition: \( 2 + 2 + 2 = 6 \)
   \( = \) Multiplication: \( 2 \times 3 = 6 \)

3. Extend: \( e \times \pi = 8.5397 \ldots \)

4. Repeated Multiplication: \( 2 \times 2 \times 2 = 8 \)
   \( = \) Powers: \( 2^3 = 8 \)

5. Extend: \( e^{\pi} = 23.141 \ldots \)

Question
Can we do this with complex numbers?
Powers of Real Numbers

Caution: Not Straightforward Even for \( \mathbb{R} \)

1. Powers not commutative:

\[ a + b = b + a, \quad a \times b = b \times a, \quad a^b \neq b^a \]

2. Not all powers always defined:

\[ 0^{-1} = ? \quad (-1)^{1/2} = ? \]

3. Not all powers always unique:

\[ 4^{1/2} = \begin{cases} 2 \\ -2 \end{cases} \]
1. $a^n$ is repeated multiplication:

$$a^n = a \times a \times \cdots \times a$$

2. Multiplication of powers is addition of exponents:

$$a^b \times a^c = a^{b+c}$$

3. Repeated powers is multiplication of exponents:

$$a^{bc} = a^{b \times c}$$
Reduction to Exponentials

Case I: Exponent is Complex

\[ a^{x+iy} = a^x a^{iy} \]  \hspace{2cm} \text{Rule II}

\[ = a^x (e^{\log a})^{iy} \]

\[ = a^x e^{iy \log a} \]  \hspace{2cm} \text{Rule III}

Conclusion: To compute \( a^z \) need to know \( e^{it} \) where \( t \in \mathbb{R} \)
Defining Exponentials

Question
How do we define $e^{it}$ for $t \in \mathbb{R}$?

How do we define $e^t$ for $t \in \mathbb{R}$?
Course Description

Exponentials via Differential Equations

Definition
For $t \in \mathbb{R}$, $e^{it}$ is the value at $t$ of the unique solution of the ODE
\[ y' = iy, \quad y(0) = 0. \]

Definition
For $t \in \mathbb{R}$, $e^{it}$ is the value at 1 of the unique solution of the ODE
\[ y' = ity, \quad y(0) = 0. \]
In Pictures

Step size: 0.01
In Symbols

To Solve

\[ z' = iz, \quad z(0) = 1, \quad z(t) \in \mathbb{C} \]

1. Write \( z(t) = x(t) + iy(t) \)
2. ODE becomes \( x'(t) + iy'(t) = i(x(t) + iy(t)) = -y(t) + ix(t) \)
3. Separate: \( x'(t) = -y(t), \quad y'(t) = x(t) \)
4. Separate further: \( x''(t) = -x(t) \),
5. Solutions: \( x(t) = a \cos(t) + b \sin(t) \),
6. Initial conditions: \( x(0) = 1 \) so \( x(t) = \cos(t) \)
7. Substitute: \( y(t) = -x'(t) = \sin(t) \)
8. Final solution: \( z(t) = \cos(t) + i\sin(t) \)
Course Description

In Symbols

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Truth By Definition

Definition

\[ e^{it} = \cos(t) + i \sin(t) \]

\[ e^{i\pi} = \cos(\pi) + i \sin(\pi) = -1 \]

Notation: Polar coordinates

\[(r, \theta) \leftrightarrow r \cos(\theta) + ir \sin(\theta) = re^{i\theta}\]
Powers

Solved:

\[ a^{x+iy} = a^x e^{iy \log a} \]

Question

What about \((x + iy)^a\)? Or \((x + iy)^{u+iv}\)?
Case II: Complex Base, Integral Exponent

\[ n \in \mathbb{Z}, \quad (x + iy)^n = ? \]

\[ n \in \mathbb{N}, (x + iy)^n = (x + iy) \times (x + iy) \times \cdots \times (x + iy) \]

Easier in polar coordinates:

\[ (r, \theta)^n \leftrightarrow (re^{i\theta})^n = r^n e^{in\theta} \]

Then to extend to \( n \in \mathbb{Z} \):

\[ (x + iy)^{-n} = \frac{1}{(x + iy)^n} \]
Case III: Complex Base, Rational Exponent

\[ q \in \mathbb{Q}, \quad (x + iy)^q = ? \]
\[ n \in \mathbb{N}, (x + iy)^{1/n} = ? \]

Solution of:

\[ ((x + iy)^{1/n})^n = x + iy \]

Easier in polar coordinates:

\[ (se^{i\varphi})^n = re^{i\theta} \]

So \( s^n = r \) and \( n\varphi = \theta \).
Complex Base

\[(re^{i\theta})^{1/n} = r^{1/n}e^{i\theta/n}\]

Ambiguous!

Not \(r^{1/n}\): \(r\) is a length, so positive real number

\[e^{2\pi i} = \cos(2\pi) + i\sin(2\pi) = 1 \text{ so } re^{i\theta} = re^{i\theta+2\pi i}\]

\[(re^{i\theta})^{1/n} = (re^{i\theta+2\pi i})^{1/n} = r^{1/n}e^{i\theta/n+2\pi i/n}\]

... and for \(4\pi i, 6\pi i, \ldots\)

Conclusion:

\[(re^{i\theta})^{1/n} = r^{1/n}e^{i\theta/n+2k\pi i/n}, \quad k \in \{0, \ldots, n - 1\}\]
Example: 4th Roots of $4 + i3$

$$r = \sqrt{4^2 + 3^2} = 5,$$

$$\theta = \tan^{-1}(\frac{4}{3}) = 0.9273$$

$$\Rightarrow 5^{\frac{1}{4}} = 1.4953,$$

$$\Rightarrow 0.23182 + n\pi/2$$
Powers

In General:

\[(x + iy)^{u+iv} = \left(re^{i\theta}\right)^{u+iv}\]

\[= r^u r^iv e^{i\theta u} e^{i\theta iv}\]

\[= r^u e^{-\theta v} e^{i(v \log(r) + \theta u)}\]

Same ambiguity: \(x + iy = re^{i\theta + 2k\pi i}\), but can be infinitely many solutions

Neat Fact

\[i^i = (e^{i\pi/2})^i = e^{i^2\pi/2} = e^{-\pi/2} \approx 0.20788 \approx \frac{1}{5}\]
Pretty Pictures

Addition, Subtraction

\[ -2 + i \]

\[ 1 + i2 \]

\[ + (3 + i) \]

\[ 3 + i \]
Pretty Pictures

Complex conjugation, real and imaginary parts
Pretty Pictures

Polar Co-ordinates
Pretty Pictures

\[ \sqrt{13} = \sqrt{3^2 + 2^2} \]

\[ \tan^{-1}(\frac{2}{3}) \]

Absolute Value: \( z = x + iy \mapsto |z| = \sqrt{x^2 + y^2} \)

Argument: \( z = x + iy \mapsto \text{arg}(z) = \tan^{-1}(\frac{y}{x}) \)
Pretty Pictures

Multiplication = Rotation and Dilation
Reciprocal: \( z \mapsto \frac{1}{z} \)
Not Just Pretty Pictures

\[ z \mapsto e^z \]
Question
Write the complex number \( w = \frac{3-i}{2i-1} \) in polar form. Find all the solutions of the equation \( z^4 = w \) and draw the solutions in the complex plane.

\[
\begin{align*}
w &= \sqrt{2}e^{i\frac{5\pi}{4}} \\
z &= re^{i\theta} \\
\implies r^4 &= \sqrt{2}, \\
4\theta &= \frac{5\pi}{4} \\
\implies r &= 2^{\frac{1}{8}}, \\
\theta &= \frac{5\pi}{16}(\frac{\pi}{2}(\frac{\pi}{2}(\frac{\pi}{2}(\frac{\pi}{2})))
\end{align*}
\]
On the Complex Plane
Summary

- Powers of complex numbers by complex numbers defineable
- Usually multi-valued
- Formula for roots (polar coordinates):
  \[ (re^{i\theta})^{1/n} = r^{1/n}e^{i\theta/n + i2p/n} \]
- For \( w \neq 0 \), \( z^n = w \) has exactly \( n \) distinct solutions
- \( e^{it} \) is unique solutions of \( z'(t) = z(t) \) with initial condition \( z(0) = 1 \)
- \( e^{it} = \cos(t) + i \sin(t) \)