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Aims and Objectives

By the end of this lecture, you will
- know what it means to solve an ODE
- have seen the method of “reduction of order”
Recap: Complex Numbers

Complex Numbers

- Extend the real numbers
- Include $i$
- Representations: $z = x + iy = re^{i\theta}$
- Addition, multiplication, division as before
- Powers and roots
- Exponential function $z(t) = e^{it}$ satisfies $z'(t) = z(t)$

Reason for Existing

Extend the language of mathematics to describe more stuff.
Typical Exam Questions

**Question**
Find a solution of the differential equation
\[ y'' - (3x^2 + 4x^{-1})y' + (3x + 4x^{-2})y = 0 \]
which is linearly independent of the solution \( y = x \).

**Question**
The motion of a mechanical system is described by the differential equation
\[ y'' + 6y' + 18y = 0. \]
Determine whether the motion is underdamped, overdamped, or if there is critical damping. Find a particular solution \( y(t) \) which satisfies the initial conditions \( y(0) = 0, \ y'(0) = 0.6 \).
**Typical Real Life Question**

**Question**
Should I buy shares in Statoil today?

**Question**
Will the share price in Statoil go up or down?

**Question**
What is the value of the solution of

\[
dX_t = \mu X_t dt + \sigma X_t dB_t
\]

at \( t = 1 \) day more or less than its value today?
What Are ODEs For?

**Question**
What is Mathematics For?

**Science**
To Observe and Effect
(More seriously)

Science : To predict events
Mathematics : To model events

**The Mathematical Model**
To describe “reality” in order to make the prediction.
Processes

Question

1. I put in $X$, what do I get?
2. I got $Y$, where did I start?
3. I put in $X$, I got $Y$, how did I get it?

ODEs

Differential Equations: a vast source of reliable models describing real processes.
What is an ODE?

Model $\leftrightarrow$ Process

**Process:** able to read off prediction

- Height of falling object: $h(t) = -gt^2$
- Displacement of pendulum: $x(t) = \sin(t)$
- Population of earth: $p(t) = e^{kt}$

**Differential Equation:** not able to “read off” prediction:

$$y'(t) = ky(t)$$

**Conclusion**

An ODE is **not** a model, it is a way of specifying a model.
What is an ODE?

**Working Definition**
An ODE is a way of specifying a function $f : \mathbb{R} \rightarrow \mathbb{R}$ by specifying a condition that its derivatives must satisfy.
(More generally: $f : I \rightarrow \mathbb{R}^n$)

**Important Consequence: I**
If you know the function, you no longer need the ODE.

**Important Consequence: II**
If you don’t know the function, you might still get useful information directly from the ODE.
Knowing When You’re Done

Solving an ODE
To solve an ODE means to find a function satisfying it.

Absolutely Vital Things to Remember

1. The method doesn’t matter, just the function
2. A function is a solution if and only if it satisfies the ODE
Second Order Homogeneous

Working Definition
An ODE is a way of specifying a function $f : \mathbb{R} \rightarrow \mathbb{R}$ by specifying a condition that its derivatives must satisfy.

Examples
1. $y' = y$
2. $y' y = 1$
3. $y'' y + y'^2 = 0$
4. $e^{y''} = \log(t)$
What Makes ODEs Difficult ... Interesting

**Big Theorem**
Under very mild conditions, solutions *always* exist!

**Big Problem**
But there is no *one* method of finding them, and no guarantee that a *closed form* exists.

**Big Consequences**

- Lots of methods developed for solving lots of different ODEs.
- Lots of methods developed for analysing solutions *even if* the solution can’t be written down.
- Leads to classifying ODEs by techniques and properties.
Simple ODEs

General Principles

1. More derivatives $\implies$ more complicated
2. More interaction $\implies$ more complicated

In order of simplicity

0. Zeroth order:
   \[ y(t) = e^t \]

1. First order:
   \[
   \begin{align*}
   y'(t) &= e^t \\
   y'(t) &= e^t y(t) \\
   y'(t) &= e^t y(t) + e^{-t}
   \end{align*}
   \]

2. Second order:
   \[
   \begin{align*}
   y''(t) &= e^t \\
   y''(t) &= e^t y'(t) \\
   y''(t) &= e^t y'(t) + e^{-t} y(t) \\
   y''(t) &= e^t y'(t) + e^{-t} y(t) + e^{2t}
   \end{align*}
   \]
Second Order Homogeneous

Standard Form

\[ y''(t) + p(t)y'(t) + q(t) = 0 \]

- Second order
- Linear
- Homogeneous

Why So Special?

\( y_1 \) and \( y_2 \) solutions \( \implies ay_2 + by_2 \) solution

know two solutions \( \implies \) know all solutions
Solving ODEs

To Solve

\[ y''(t) + p(t)y'(t) + q(t) = 0 \]

No general solution so need more information.

Two Types

- Information about \( p(t) \) and \( q(t) \)
- Information about \( y(t) \)
- Looking for two solutions, so what if we knew one?
Solving Polynomials

What are the roots of

\[ x^5 - 15x^4 + 85x^3 - 225x^2 + 274x - 120 \]
given that four of them are 1, 2, 3, 4?

What are the roots of

\[ x^2 - 2x = x(x - 2) \]

What are the roots of

\[ x^2 - 7x + 12 = (x - 3)^2 - (x + 3) = (x - 3)((x - 3) - 1) \]
given that one of them is 3?
Reduction of Order

Basic Principle
Given one solution, write the other in terms of the first and rearrange to get a simpler condition.

Most Important Point

1. Substitute \( y = uy_1 \)
2. Solve for \( u' \)
Question

Find a solution of the differential equation
\[ y'' - (3x^2 + 4x^{-1})y' + (3x + 4x^{-2})y = 0 \]
which is linearly independent of the solution \( y = x \).

1. Write \( y = uy_1 \)

\[ y = ux \]

2. Solve for \( u' \)
\[ y'' - (3x^2 + 4x^{-1})y' + (3x + 4x^{-2})y = 0 \]

\[ (ux)'' - (3x^2 + 4x^{-1})(ux)' + (3x + 4x^{-2})(ux) = 0 \]

\[ u'' x + 2u' - (3x^2 + 4x^{-1})(u' x + u) + (3x + 4x^{-2})(ux) = 0 \]

\[ u'' x + (2 - 3x^3 - 4)u' + (-3x^2 - 4x^{-1} + 3x^2 + 4x^{-1})u = 0 \]

\[ u'' x - (2 + 3x^3)u' = 0 \]

\[ U' x - (2 - 3x^3)U = 0 \]

\[ \frac{U'}{U} = 3x^2 + 2x^{-1} \]

\[ \log U = x^3 + 2 \log x \]

\[ U = x^2 e^{-x^3} \]

\[ u' = x^2 e^{-x^3} \]

\[ u = \frac{1}{3} e^{-x^3} \]

\[ y = \frac{1}{3} x e^{-x^3} \]
Why It Works

- General nonsense \implies \text{almost any two solutions enough}
- Not enough only if have \( k \in \mathbb{R} \) such that \( y_1 = ky_2 \)
  \( (y_1 = ky_2 \iff \text{“linearly dependent”}; \text{if not, “linearly independent”}) \)
- General nonsense \implies \( y_1(t) \) almost always not zero
- \implies \text{for a fixed} \( t \), have \( k \in \mathbb{R} \) with \( y_2(t) = ky_1(t) \)
- But \( k \) can vary with \( t \); function, not constant; call it \( u(t) \)
- ODE specifies \( y_1 \) and \( y_2 \), so also \( u \), so by rearranging the ODE we can get something that \( u \) must satisfy
- But the ODE doesn’t know we already know \( y_1 \)!
  So \( u = \text{const} \) satisfies this new ODE, so ODE must be
  \[ f(t)u''(t) + \tilde{p}(t)u'(t) = 0. \]
  First order in \( u' \)
- So solve for \( u' \), then for \( u \), and finally for \( y_2 \)
Summary

- ODEs form basis of large range of mathematical models
- Solving ODEs is an art
- Second order linear homogeneous ODEs are complicated enough to be useful but simple enough to be analysed
- If one solution is known, use reduction of order to find another