Lecture 13: Inverting Matrices

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Aims and Objectives

By the end of this lecture, you will

► know how to characterise invertible matrices
► know how to invert a matrix using G E
► know how to compute determinants using G E
► know when to invert a matrix and when not to
Recap

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\
    a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\
    a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3
\end{align*}
\]

\[
\begin{bmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix}
= \begin{bmatrix}
    b_1 \\
    b_2 \\
    b_3
\end{bmatrix}
\]

\[\mathbf{x} \mapsto \mathbf{A}\mathbf{x}\] is a process

\[\text{Composition of processes} \quad \longrightarrow \quad \mathbf{A}\mathbf{B}\]

\[
\begin{bmatrix}
    1 & 2 \\
    3 & 4
\end{bmatrix}
\begin{bmatrix}
    1 & 2 & 3 \\
    4 & 5 & 6
\end{bmatrix}
= \begin{bmatrix}
    9 & 12 & 15 \\
    19 & 26 & 33
\end{bmatrix}
\]

\[\text{Juxtaposition of processes} \quad \longrightarrow \quad \mathbf{A} + \mathbf{B}\]

\[\text{Operations:} \quad \mathbf{A}\mathbf{B}, \quad \mathbf{A} + \mathbf{B}, \quad \lambda \mathbf{A}, \quad \mathbf{I}_n, \quad 0_{m,n}\]

\[\text{Usual rules except} \quad \mathbf{A}\mathbf{B} \quad \text{not necessarily} \quad \mathbf{B}\mathbf{A}\]
Invertible Processes

Invertible

A process is invertible if:
1. Each input is uniquely determined by its output
2. Each potential output is possible

Remark

1. A process is invertible $\iff$ it can be reversed
2. There is another process which reconstructs the input from the output
Invertible Processes

Going all the way around gets you right back where you started.
Invertible Processes

Definition

A process is said to be invertible if

1. each input is uniquely determined by its output, and
2. each potential output is possible.

A matrix is invertible if it represents an invertible process

1. Each input is uniquely determined by its output
   \( Ax = b \) has at most one solution, so \( m \geq n \)
2. Each potential output is possible
   \( Ax = b \) has at least one solution, so \( m \leq n \)

Invertible matrix \( \implies \) square
Invertible Matrices

Lemma

\( A \) is invertible \( \iff \exists B : AB = I_m, BA = I_n \)

\[
I_n := \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix}
\]

Note

Inverses are necessarily unique:

\[
B = BI_m = B(AC) = (BA)C = I_n C = C
\]
For a square matrix $Ax = b$

always has a solution $\iff$ solutions are unique

$\iff$

invertible
Why Linear Systems?

Recall

\[ y'' + p(t)y' + q(t)y = r(t) \]

General solution: \( y = y_p + ay_1 + by_2 \)

**Linearity** \( \implies \) construct all solutions from a handful

**Key Step**

\( y_1 \) and \( y_2 \) solutions of homogeneous

\( ay_1 + by_2 \) solutions of homogeneous
Linearity Tools

Theorem

Matrices work just the same.

Question

\[ x_1 + 2x_2 + 3x_3 = 2 \]
\[ 4x_1 + 5x_2 + 6x_3 = 5 \]

has solution \( x_1 = 1, \ x_2 = -1, \ x_3 = 1 \)

\[ x_1 + 2x_2 + 3x_3 = 2 \]
\[ 4x_1 + 5x_2 + 6x_3 = 2 \]

has solution \( x_1 = -1, \ x_2 = 0, \ x_3 = 1 \)

What is a solution of \[
\begin{align*}
4x_1 + 5x_2 + 6x_3 &= 7 \\
x_1 + 2x_2 + 3x_3 &= 4
\end{align*}
\]

\( x_1 = 1 - 1 = 0, \ x_2 = -1 + 0 = -1, \ x_3 = 1 + 1 = 2 \)
Linearity

Recap: The Rules

\[ A(B + C) = AB + AC, \quad A(\lambda B) = \lambda(AB) \]

Column vectors are matrices so \( Ax \) obeys the same rules:

\[ A(x + \lambda y) = Ax + \lambda Ay \]

Consequences

1. If \( Ax = b_1 \) and \( Ax = b_2 \) have solutions, so does \( Ax = b_1 + \lambda b_2 \)

2. If \( x_1 \) and \( x_2 \) are solutions of \( Ax = b \) then \( A(x_1 - x_2) = 0 \)

3. If \( x_1 \) is a solution of \( Ax = b \) and \( x_2 \) of \( Ax = 0 \) then \( A(x_1 + x_2) = b \)
Linearity and Number of Solutions

Question
Solve

\[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
=
\begin{bmatrix}
0 \\
3 \\
6 \\
\end{bmatrix}
\]
Solution

Method

\[
\begin{bmatrix}
1 & 2 & 3 & 0 \\
4 & 5 & 6 & 3 \\
7 & 8 & 9 & 6 \\
\end{bmatrix}
\]

\[R_2 \leftrightarrow R_2 - 4R_1\]

\[
\begin{bmatrix}
1 & 2 & 3 & 0 \\
0 & -3 & -6 & 3 \\
7 & 8 & 9 & 6 \\
\end{bmatrix}
\]

\[R_3 \leftrightarrow R_3 - 7R_1\]

\[
\begin{bmatrix}
1 & 2 & 3 & 0 \\
0 & -3 & -6 & 3 \\
0 & -6 & -12 & 6 \\
\end{bmatrix}
\]

\[R_3 \leftrightarrow R_3 - 2R_2\]

\[
\begin{bmatrix}
1 & 2 & 3 & 0 \\
0 & -3 & -6 & 3 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[R_2 \leftrightarrow -\frac{1}{3}R_2\]

\[
\begin{bmatrix}
1 & 2 & 3 & 0 \\
0 & 1 & 2 & -1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
Conclusions

\[
\begin{bmatrix}
1 & 2 & 3 & b_1 \\
0 & 1 & 2 & -\frac{1}{3}(b_2 - 4b_1) \\
0 & 0 & 0 & b_3 - 2b_2 + b_1 \\
\end{bmatrix}
\]

1. There is a solution \( \iff b_3 - 2b_2 + b_1 = 0 \)
2. The number of “free variables” is always the same.

Conclusion
If \( Ax = b \) has a solution, the number of solutions is independent of \( b \).

\( \left( \text{at the moment “number of solutions” means 1 or } \infty! \right) \)
Best Invertibility Test

Current Invertibility Tests
For square $A$, the following are equivalent:

1. $A$ represents an invertible process
2. There is a matrix $B$ (same size as $A$) with $AB = I_n = BA$
3. There is a matrix $B$ (same size as $A$) with $BA = I_n$
4. For every $b$, if $Ax = b$ has a solution it is unique
5. $Ax = 0$ has only one solution ($x = 0$)
6. For every $b$, $Ax = b$ has at least one solution

Crucial Facts

1. If $Ax = b$ has a solution, the number is independent of $b$
2. $Ax = 0$ has a solution ($x = 0$)
Computing the Inverse

Linearity also helps with finding the inverse

Goal

Find $B$ such that $AB = I_n$

Introducing ... the elementary vectors:

$$e_j := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

$j$th row
Splitting a Vector

Key Fact

Any column vector is a sum of scaled elementary vectors.

\[
\begin{bmatrix}
2 \\
\pi \\
e
\end{bmatrix}
= 2 \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} + \pi \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix} + e \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\]

Consequence

\(Ax = e_j\) solution for all \(j\) \(\iff\) \(Ax = b\) solution for all \(b\)
Computing the Inverse Strategy

Solve \( Ax = e_j \) simultaneously!

\[
\begin{bmatrix}
1 & 2 & 3 & 1 & 0 & 0 \\
2 & 3 & 4 & 0 & 1 & 0 \\
3 & 4 & 6 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{array}{c}
R_2 - 2R_1 \\
R_3 - 3R_1 \\
\end{array}
\begin{bmatrix}
1 & 2 & 3 & 1 & 0 & 0 \\
0 & -2 & -2 & -2 & 1 & 0 \\
0 & -2 & -3 & -3 & 0 & 1 \\
\end{bmatrix}
\begin{array}{c}
R_3 - 2R_2 \\
R_1 + 2R_2 \\
\end{array}
\begin{bmatrix}
1 & 0 & -1 & -3 & 2 & 0 \\
0 & -1 & -2 & -2 & 1 & 0 \\
0 & 0 & 1 & 1 & -2 & 1 \\
\end{bmatrix}
\begin{array}{c}
R_2 + 2R_3 \\
R_1 + R_3 \\
\end{array}
\begin{bmatrix}
1 & 0 & 0 & -2 & 0 & 1 \\
0 & -1 & 0 & 0 & -3 & 2 \\
0 & 0 & 1 & 1 & -2 & 1 \\
\end{bmatrix}
\begin{array}{c}
R_2 \rightarrow -R_2 \\
\end{array}
\begin{bmatrix}
1 & 0 & 0 & -2 & 0 & 1 \\
0 & 1 & 0 & 0 & 3 & -2 \\
0 & 0 & 1 & 1 & -2 & 1 \\
\end{bmatrix}
\]
Check Your Answer

\[
\begin{bmatrix}
-2 & 0 & 1 \\
0 & 3 & -2 \\
1 & -2 & 1
\end{bmatrix}
\]

is the inverse for \( A \).

Check: \( AB = I_3 \) and \( BA = I_3 \)

Summary

1. To answer “Is \( A \) invertible?”, look for solutions to \( Ax = b \).
2. To answer “What is the inverse of \( A \)?”, apply full Gaussian Elimination to \( \begin{bmatrix} A & I_n \end{bmatrix} \)

Note: Can do this without knowing that \( A \) is invertible.
Is It Invertible?

Which of the following are invertible?

\[
\begin{bmatrix}
1 & 2 \\
2 & 4
\end{bmatrix}
\quad 
\begin{bmatrix}
1 & 2 \\
2 & 3
\end{bmatrix}
\quad 
\begin{bmatrix}
1 & 0 \\
2 & 1
\end{bmatrix}
\quad 
\begin{bmatrix}
0 & 2 \\
2 & 1
\end{bmatrix}
\quad 
\begin{bmatrix}
0 & 2 \\
2 & 3
\end{bmatrix}
\quad 
\begin{bmatrix}
0 & 0 \\
2 & 1
\end{bmatrix}
\]
Is It Invertible?

Question

When is \[
\begin{bmatrix}
  a & b \\
  c & d \\
\end{bmatrix}
\] invertible?

How many solutions are there of:

\[
\begin{align*}
ax + by &= 0 \\
(cx + dy) &= 0
\end{align*}
\]

1. \(a \neq 0\):

\[
ax + by = 0 \\
(d - c^{b/a})y = 0
\]

Non-trivial solution \(\Longleftrightarrow d - c^{b/a} = 0\)

2. \(a = 0\):

2.1 \(c = 0 \implies\) non-trivial solution \((x \text{ anything})\)

2.2 \(b = 0 \implies\) non-trivial solution \((cx + dy = 0)\)
Is It Invertible?

\[ ax + by = 0 \]
\[ cx + dy = 0 \]

has non-trivial solutions if:

1. \( a \neq 0 \) and \( d - \frac{c}{a}b = 0 \)
2. \( a = 0 \) and either \( b = 0 \) or \( c = 0 \)

Non-trivial solutions \( \iff \) \( ad - bc = 0 \)

In Addition

Inverse is:

\[
\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
\]
Defining Determinants

Definition

Let $A$ be a square matrix, the determinant of $A$ is a number, $\det(A)$, defined by the properties:

1. $\det(I_n) = 1$
2. $\det(AB) = \det(A) \det(B)$
3. $\det(A^{-1}) = (\det(A))^{-1}$
4. $\det\begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} = \lambda_1 \lambda_2 \ldots \lambda_n$
Uses of Determinants

None

Allegedly

\[ A \text{ invertible} \iff \det(A) \neq 0 \]

but \textit{practically} useless
Computing Determinants

Rules for Computation

1. Row Swaps: if $B$ and $A$ differ by a row swap

   $$\det(A) = -\det(B)$$

2. Row Scales: if $B$ and $A$ differ by a row scale

   $$\det(A) = \lambda \det(B)$$

3. Row Adds: if $B$ and $A$ differ by a row addition

   $$\det(A) = \det(B)$$

4. s/row/column/i
Computing Determinants

Remember:
\[
\begin{vmatrix}
0 & 1 \\
2 & 1 \\
\end{vmatrix} = -2
\]

Strategy: Use Gaussian Elimination, keeping track of number of row swaps and row scalings

\[
\begin{vmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
\end{vmatrix} = \begin{vmatrix}
0 & -3 & -6 \\
0 & -6 & -12 \\
\end{vmatrix}
\]

\[
\begin{array}{c}
(R_3 \rightarrow R_3 - 7R_1) \\
(R_2 \rightarrow R_2 - 4R_1)
\end{array}
\]

\[
\begin{vmatrix}
1 & 2 & 3 \\
1 & 2 & 3 \\
\end{vmatrix} = \begin{vmatrix}
1 & 2 & 3 \\
0 & 1 & 2 \\
0 & 0 & 0 \\
\end{vmatrix}
\]

\[
\begin{array}{c}
(R_2 \rightarrow -\frac{1}{3}R_2) \\
(R_3 \rightarrow R_3 + 6R_2)
\end{array}
\]

\[
= \begin{vmatrix}
1 & 2 & 3 \\
0 & 1 & 2 \\
0 & 0 & 0 \\
\end{vmatrix} = 0
\]
Computing Determinants

\[
\begin{vmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 10
\end{vmatrix}
= \begin{vmatrix}
1 & 2 & 3 \\
0 & -3 & -6 \\
0 & -6 & -11
\end{vmatrix}
\]

\[
= -3 \begin{vmatrix}
0 & 1 & 2 \\
0 & -6 & -11
\end{vmatrix}
\]

\[
= -3 \begin{vmatrix}
0 & 1 & 2 \\
0 & 0 & 1
\end{vmatrix}
\]

\[
= -3
\]
Computing Determinants

\[
\begin{vmatrix}
0 & 1 & 2 \\
2 & 4 & 6 \\
3 & 5 & 6 \\
\end{vmatrix} = \begin{vmatrix}
2 & 4 & 6 \\
0 & 1 & 2 \\
3 & 5 & 6 \\
\end{vmatrix} = -2
\]

\[
\begin{vmatrix}
1 & 2 & 3 \\
2 & 4 & 6 \\
3 & 5 & 6 \\
\end{vmatrix} = \begin{vmatrix}
1 & 2 & 3 \\
0 & 1 & 2 \\
0 & -1 & -3 \\
\end{vmatrix} = -2
\]

\[
\begin{vmatrix}
1 & 2 & 3 \\
2 & 4 & 6 \\
3 & 5 & 6 \\
\end{vmatrix} = \begin{vmatrix}
1 & 2 & 3 \\
0 & 0 & 1 \\
0 & 0 & -1 \\
\end{vmatrix} = 2
\]
A square matrix is invertible if

1. $A$ represents an invertible process
2. there is a matrix $B$ (same size as $A$) with $AB = I_n$
3. there is a matrix $B$ (same size as $A$) with $BA = I_n$
4. for every $b$, if $Ax = b$ has a solution it is unique
5. $Ax = 0$ has only one solution ($x = 0$)
6. for every $b$, $Ax = b$ has at least one solution
7. $\det A \neq 0$
Using Invertibility

Use?
To solve $Ax = b$, invert $A$: $x = A^{-1}b$
No! Need to use $GE$ to find $A^{-1}$ so just use $GE$ to solve.

Use
To solve $Ax = b$, find invertible $B$ and solve $BAx = Bb$.

Invertible $\iff$ solutions of $Ax = b$ are the same as solutions of $BAx = Bb$

This is basically what G E is all about.
Use Gaussian elimination for everything

Invertible: solve $Ax = 0$

Inverse: apply G E to $\begin{bmatrix} A & I_n \end{bmatrix}$ to try to get $\begin{bmatrix} I_n & A^{-1} \end{bmatrix}$

Determinants: compute using G E