Today's List:

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Closed Subspaces

•) $W \subseteq H$, W subspace and closed

$\{b_k\} \subseteq W$ orthonormal

then $\sum \lambda_k b_k$ exists in W

iff $\sum |\lambda_k|^2 < \infty$

Least Squares : Closest points in subspaces

Two Methods

•) Use an orthonormal basis

$W \subseteq V$, $\{b_n\}$ for W o.n. basis

complete

$v \in V$, $w_v \in W$ closest

$w_v = \sum_n \langle w_v, b_n \rangle b_n$

Trick: $v - w_v \perp b_n$, $\langle v, b_n \rangle = \langle w_v, b_n \rangle$

$w_v = \sum_n \langle v, b_n \rangle b_n$

•) Use the " $A^TAx = A^Tb$ " method

Closest point is in W,
find a linear transformation $A: \mathbb{R}^n \to V$
such that $\operatorname{im} A = W$

$w_v = Ax$ for some $x \in \mathbb{R}^n$

$v - w_v$ stuff to forget
this is orthogonal to im A

$\Rightarrow \quad A^T(v - w_v) = 0$

$[\langle a_i, v - w_v \rangle] = 0$

$A^Tv = A^Tw_v$
Write $w_v = Ax$
and solve $A^TAx = A^Tv$ (GE)
answer: Ax

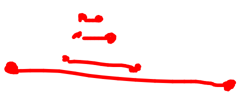
$$a_i \perp v - w_v$$

$$\langle a_i, v - w_v \rangle = 0$$

$$a_i^T (v - w_v) = 0$$

$$\begin{bmatrix} a_i^T (v - w_v) \end{bmatrix} = \underline{0}$$

$$A^T (v - w_v)$$



(M,d) non-empty complete, $T: M \to M$ contraction

$\alpha \in (0,1)$ st $d(T(x), T(y)) \leq \alpha\, d(x,y)$

then $\exists!\, x^* \in M$ st $T(x^*) = x^*$

- start with x_1, define $x_n = T(x_{n-1})$
- T contraction $\Rightarrow$ (x_n) Cauchy
- Complete $\Rightarrow$ (x_n) converges say to x^*
- T contraction $\Rightarrow$ T cts $\Rightarrow$ $T(x^*) = \lim T(x_n)$

$$= \lim (x_{n-1})$$

$$= x^*$$

ODE

$y' = F(y) \qquad y(0) = a,$

$$y(t) = a + \int_0^t F(y(s))\,ds$$

$$T : C([0,r], \mathbb{R}) \longrightarrow C([0,r], \mathbb{R})$$

$$T(y)(t) = a + \int_0^t F(y(s))\,ds$$

Need conditions to ensure T is a contraction

- $F : \mathbb{R} \to \mathbb{R}$ is itself a contraction, $r = 1$