

TMA4145 Linear Methods

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Lecture 28: Revision I: Summary of Summaries

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Table of Categories

Category	Objects	Morphisms
MSp	(M, d)	continuous, Lipschitz, pointwise continuous
VSp	$(V, +, 0, \lambda)$	linear
NVSp	$(V, \ \cdot\)$	continuous + linear
BSp	Banach space	continuous + linear
IPSp	$(V, \langle \cdot, \cdot \rangle)$	continuous + linear, isometry
HSp	Hilbert space	continuous + linear, isometry

Use of Categories

MSp Home of sequences, place for convergence

VSp Linear \implies decomposable

NVSp Combine linearity and convergence

BSp NVSp + nice limit behaviour

IPSp Home of geometry

HSp All of the above!

Metric Spaces

Purpose

Place to discuss convergence

Examples

$(\mathbb{R}, |\cdot|)$, $(S^n, \|p - q\|_2)$, $(\mathbb{N}_0, |1/n - 1/m|)$

Key Technologies

- ▶ Sequences — convergent and Cauchy
- ▶ Neighbourhoods

Key Theorem

Banach's Fixed Point Theorem

Metric Spaces

Typical Questions

1. Verify:
 - 1.1 (M, d) is a (complete) metric space
 - 1.2 (x_n) is a convergent/Cauchy sequence
 - 1.3 $f : M \rightarrow N$ is a continuous function
 - 1.4 $T : M \rightarrow M$ is a contraction
2. Use:
 - 2.1 Properties of limit from sequence
 - 2.2 Fixed point theorem
3. Extend:
 - 3.1 Use one sequence to test another
 - 3.2 Use functions to test sequences and vice versa

Vector Spaces

Purpose

Linearity means **splits nicely**

Examples

\mathbb{C}^n , Poly_k

Key Technologies

- ▶ Linear Transformations — matrices
- ▶ Factorisations

Key Meta Theorem

Matrices factor into **nice** pieces.

Vector Spaces

Typical Questions

1. Verify:
 - 1.1 V is a vector space
 - 1.2 $T : V \rightarrow W$ is a linear transformation
2. Use:
 - 2.1 Solve $Tx = b$
 - 2.2 Factorise A as LU , QR , or $U\Sigma V^*$
3. Extend:
 - 3.1 Use properties of factorisations
 - 3.2 Transfer from matrices to transformations and back

Normed Vector Spaces/Banach Spaces

Purpose

Combines **convergence** with **linearity**

Examples

$\ell^0, \ell^1, \ell^2, \ell^\infty, c_0, C([0, 1], \mathbb{C}), L^1(0, 1), L^2(0, 1)$

Key Technologies

- ▶ Sequences + Linear Transformations
- ▶ Series

Key Theorem

... *not really had one* ...

Normed Vector Spaces/Banach Spaces

Typical Questions

1. Verify:
 - 1.1 $(V, \|\cdot\|)$ is a normed vector space
 - 1.2 $(V, \|\cdot\|)$ is a Banach space
 - 1.3 $T: V \rightarrow W$ is a continuous linear transformation
2. Use:
 - 2.1 Combine techniques of metric and vector spaces

Inner Product Spaces/Hilbert Spaces

Purpose

Combines linear, convergence, and geometry

Examples

$\ell^2, L^2(0, 1)$

Key Technologies

- ▶ Orthogonality
- ▶ Duality

Key Theorems

- ▶ Closest Point
- ▶ Riesz Representation

Inner Product Spaces/Hilbert Spaces

Typical Questions

1. Verify:
 - 1.1 $(H, \langle \cdot, \cdot \rangle)$ is a Hilbert space
 - 1.2 B is an orthonormal basis/orthogonal family
2. Use:
 - 2.1 Orthogonalise a sequence
 - 2.2 Find a closest point
 - 2.3 Minimise a length
 - 2.4 Use adjoints
3. Extend:
 - 3.1 Work with unusual inner products
 - 3.2 Transfer knowledge from finite to infinite dimensions

The Final Diagram

