TMA4145 Linear Methods

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Lecture 28: Revision I: Summary of Summaries

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Table of Categories

Category	Objects	Morphisms
MSp	(M, d)	continuous, Lipschitz, pointwise continuous
VSp	$(V, +, 0, \lambda)$	linear
NVSp	$(V, \|\cdot\|)$	continuous + linear
BSp	Banach space	continuous + linear
IPSp	$(V, \langle\cdot,\cdot\rangle)$	continuous + linear, isometry
HSp	Hilbert space	continuous + linear, isometry

Use of Categories

MSp	Home of sequences, place for convergence
VSp	Linear $\implies$ decomposable
NVSp	Combine linearity and convergence
BSp	NVSp $+$ nice limit behaviour
IPSp	Home of geometry
HSp	All of the above!

Metric Spaces

Purpose

Place to discuss convergence

Examples

$(\mathbb{R}, |\cdot|)$, $(S^n, \|p - q\|_2)$, $(\mathbb{N}_0, |{}^1\!/\!_n - {}^1\!/\!_m|)$

Key Technologies

- Sequences — convergent and Cauchy
- Neighbourhoods

Key Theorem

Banach's Fixed Point Theorem

Metric Spaces

Typical Questions

1. Verify:
    1.1 (M, d) is a (complete) metric space
    1.2 (x_n) is a convergent/Cauchy sequence
    1.3 $f : M \to N$ is a continuous function
    1.4 $T : M \to M$ is a contraction
2. Use:
    2.1 Properties of limit from sequence
    2.2 Fixed point theorem
3. Extend:
    3.1 Use one sequence to test another
    3.2 Use functions to test sequences and vice versa

Vector Spaces

Purpose

Linearity means splits nicely

Examples

$\mathbb{C}^n$, Poly_k

Key Technologies

- Linear Transformations — matrices
- Factorisations

Key Meta Theorem

Matrices factor into nice pieces.

Vector Spaces

Typical Questions

1. Verify:
   1.1 V is a vector space
   1.2 $T : V \to W$ is a linear transformation
2. Use:
   2.1 Solve $T\mathrm{x} = \mathrm{b}$
   2.2 Factorise A as LU, QR, or $U\Sigma V^*$
3. Extend:
   3.1 Use properties of factorisations
   3.2 Transfer from matrices to transformations and back

Normed Vector Spaces/Banach Spaces

Purpose

Combines convergence with linearity

Examples

ℓ^0, ℓ^1, ℓ^2, ℓ^∞, c_0, $C([0,1],\mathbb{C})$, $L^1(0,1)$, $L^2(0,1)$

Key Technologies

- Sequences + Linear Transformations
- Series

Key Theorem

... not really had one ...

Normed Vector Spaces/Banach Spaces

Typical Questions

1. Verify:
   1.1 $(V, \|\cdot\|)$ is a normed vector space
   1.2 $(V, \|\cdot\|)$ is a Banach space
   1.3 $T : V \to W$ is a continuous linear transformation
2. Use:
   2.1 Combine techniques of metric and vector spaces

Inner Product Spaces/Hilbert Spaces

Purpose

Combines linear, convergence, and geometry

Examples

ℓ^2, $L^2(0,1)$

Key Technologies

- Orthogonality
- Duality

Key Theorems

- Closest Point
- Riesz Representation

Inner Product Spaces/Hilbert Spaces

Typical Questions

1. Verify:
   1.1 $(H, \langle \cdot, \cdot \rangle)$ is a Hilbert space
   1.2 B is an orthonormal basis/orthogonal family
2. Use:
   2.1 Orthogonalise a sequence
   2.2 Find a closest point
   2.3 Minimise a length
   2.4 Use adjoints
3. Extend:
   3.1 Work with unusual inner products
   3.2 Transfer knowledge from finite to infinite dimensions

The Final Diagram