Example: Diagonal Operator

2. $D\colon \ell^2 \to \ell^2\colon D(x_1, x_2, x_3, \dots) = (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, \dots)$

$$
\begin{aligned}
\langle D^\star(y_n), (x_n)\rangle &= \langle (y_n), D(x_n)\rangle \\
&= \langle (y_n), (\lambda_n x_n)\rangle \\
&= \sum_{n\geq 1} y_n \overline{\lambda_n x_n} \\
&= \sum_{n\geq 1} \overline{\lambda_n} y_n \overline{x_n} \\
&= \langle (\overline{\lambda_n} y_n), (x_n)\rangle
\end{aligned}
$$

Example: Matrix Operator

3. $A \in \mathrm{Mat}_n(\mathbb{C})$

$$
\begin{aligned}
\langle A^\star \vec{x}, \vec{y} \rangle &= \langle \vec{x}, A\vec{y} \rangle \\
&= \vec{x}^\top \overline{A\vec{y}} \\
&= \left(\overline{A}^\top \vec{x}\right)^\top \overline{\vec{y}} \\
&= \langle \overline{A}^\top \vec{x}, \vec{y} \rangle
\end{aligned}
$$

$x^T \bar{A} \bar{y} = z^T \bar{y}$

$x^T \bar{A} = (\bar{A}^T x)^T$

$z = \bar{A}^T x$ then $z^T \bar{y} = x^T \bar{A} \bar{y}$

Proof: Quite Easy Part

- A normal
- A has an eigenvector v, eigenvalue λ
- So $(A - \lambda I)v = 0$. Thus

$$
\begin{aligned}
0 &= \left\|(A - \lambda I)v\right\|^2 \\
&= \langle (A - \lambda I)v, (A - \lambda I)v \rangle && \text{adjoint property} \\
&= \langle (A^\star - \bar{\lambda} I)(A - \lambda I)v, v \rangle && \text{normality} \\
&= \langle (A - \lambda I)(A^\star - \bar{\lambda} I)v, v \rangle && \text{adjoint property} \\
&= \langle (A^\star - \bar{\lambda} I)v, (A^\star - \bar{\lambda} I)v \rangle \\
&= \left\|(A^\star - \bar{\lambda} I)v\right\|^2
\end{aligned}
$$