TMA4145 Linear Methods

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Trondheim

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Lecture 28: Adroit Adjunctions Again And Adieu

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Recap

Definition

H Hilbert space
$T : H \to H$ continuous linear.
The adjoint of T is
$T^\star : H \to H$
satisfying

$$\langle u, T^\star v\rangle = \langle Tu, v\rangle$$

self-adjoint:	$T = T^\star$
skew-adjoint:	$T = -T^\star$
unitary:	$U^{-1} = U^\star$
normal:	$TT^\star = T^\star T$

Key Point

- Make the most of coincidences!

Examples

1. $S : \ell^2 \to \ell^2$ shift operator:
   $S(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots)$

Examples

1. $S\colon \ell^2 \to \ell^2$ shift operator:
   $S(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots)$

$$
\begin{aligned}
\langle S^\star(y_n), (x_n)\rangle &= \langle (y_n), S(x_n)\rangle \\
&= \langle (y_n), (x_{n-1})\rangle \\
&= \sum_{n\geq 2} y_n \overline{x_{n-1}} \\
&= \sum_{n\geq 1} y_{n+1} \overline{x_n} \\
&= \langle (y_{n+1}), (x_n)\rangle
\end{aligned}
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$$

So $S^\star(y_1, y_2, y_3, \dots) = (y_2, y_3, \dots)$

Shift Operator

Then

$$SS^\star(x_1, x_2, x_3, \dots) = (0, x_2, x_3, \dots)$$
$$S^\star S(x_1, x_2, x_3, \dots) = (x_1, x_2, x_3, \dots)$$

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$$S^\star S(x_1, x_2, x_3, \dots) = (x_1, x_2, x_3, \dots)$$

So not normal.

Example: Diagonal Operator

2. $D : \ell^2 \to \ell^2$: $D(x_1, x_2, x_3, \ldots) = (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, \ldots)$

Example: Diagonal Operator

2. $D\colon \ell^2 \to \ell^2\colon D(x_1, x_2, x_3, \dots) = (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, \dots)$

$$\begin{aligned}
\langle D^\star(y_n), (x_n)\rangle &= \langle (y_n), D(x_n)\rangle \\
&= \langle (y_n), (\lambda_n x_n)\rangle \\
&= \sum_{n\geq 1} y_n \overline{\lambda_n x_{n-1}} \\
&= \sum_{n\geq 1} \overline{\lambda_n} y_n \overline{x_n} \\
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$$DD^\star(x_1, x_2, x_3, \dots) = (|\lambda_1|^2 x_1, |\lambda_2|^2 x_2, |\lambda_3|^2 x_3, \dots)$$
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So

Self-adjoint	$\iff$	λ_n	$\in$	$\mathbb{R}$
Skew-adjoint	$\iff$	λ_n	$\in$	$i\mathbb{R}$
Unitary	$\iff$	λ_n	$\in$	S^1
Always normal				

Example: Matrix Operator

3. $A \in \mathrm{Mat}_n(\mathbb{C})$

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$$\begin{aligned}
\langle A^\star \vec{x}, \vec{y} \rangle &= \langle \vec{x}, A\vec{y} \rangle \\
&= \overline{\vec{x}}^\top A \vec{y} \\
&= \overline{\overline{A}^\top \vec{x}}^\top \vec{y} \\
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Example: Matrix Operator

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So $A^\star = \overline{A}^\top$

Example: Multiplication

4. $f \in L^2(0,1)$, $M_f\colon L^2(0,1) \to L^2(0,1)$ by $M_f g = fg$.

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$$
\begin{aligned}
\langle M_f^\star g, h \rangle &= \langle g, M_f h \rangle \\
&= \int_0^1 g(t)\overline{(M_f h)}(t)dt \\
&= \int_0^1 g(t)\overline{f(t)h(t)}dt \\
&= \int_0^1 \overline{f(t)}g(t)\overline{h(t)}dt \\
&= \int_0^1 (M_{\bar{f}}\, g)(t)\overline{h(t)}dt \\
&= \langle M_{\bar{f}}\, g, h \rangle
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Example: Multiplication

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&= \int_0^1 \overline{f(t)}g(t)\overline{h(t)}\,dt \\
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&= \langle M_{\bar{f}}\, g, h \rangle
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So $M_f{}^\star = M_{\bar{f}}$

Example: Multiplication

Then

$$M_f M_f{}^\star = M_{|f|^2}$$
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Note similarity to diagonal operators! Not a coincidence.

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- Diagonal $\Longrightarrow$ Normal

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Theorem (Spectral Theorem in Finite Dimensions)

$A \in \mathrm{Mat}_n(\mathbb{C})$ is normal

$$\Longleftrightarrow$$

it is orthogonally diagonalisable

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Theorem (Spectral Theorem in Finite Dimensions)

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$$\Longleftrightarrow$$

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Remark: Real version was $A = A^\top$

Proof: Very Easy Part

- A orthogonally diagonalisable

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- And $A^\star = (UDU^\star)^\star = (U^\star)^\star D^\star U^\star = U\overline{D}U^\star$

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$$AA^\star = UDU^\star U\overline{D}U^\star = UD\overline{D}U^\star$$
$$A^\star A = U\overline{D}U^\star UDU^\star = U\overline{D}DU^\star$$

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- But D diagonal so $D\overline{D} = \overline{D}D$
- Hence A normal.

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$$
\begin{aligned}
0 &= \left\|(A - \lambda I)v\right\|^2 \\
&= \langle (A - \lambda I)v, (A - \lambda I)v \rangle \\
&= \langle (A^\star - \bar{\lambda} I)(A - \lambda I)v, v \rangle \\
&= \langle (A - \lambda I)(A^\star - \bar{\lambda} I)v, v \rangle \\
&= \langle (A^\star - \bar{\lambda} I)v, (A^\star - \bar{\lambda} I)v \rangle \\
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&= \left\|(A^\star - \bar{\lambda} I)v\right\|^2
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$$

So v eigenvector of $A^\star$, eigenvalue $\bar{\lambda}$.

Quite Easy Part

- Extend to orthonormal basis $\{v, u_2, \ldots, u_n\}$

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- Work out matrix of A:

$$\langle Av, v \rangle = \langle \lambda v, v \rangle = \lambda$$
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- A_1 also normal, so continue.

Example

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

is unitary, hence normal.

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$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$$

Spectral Theory

1. Quite dull in finite dimensions

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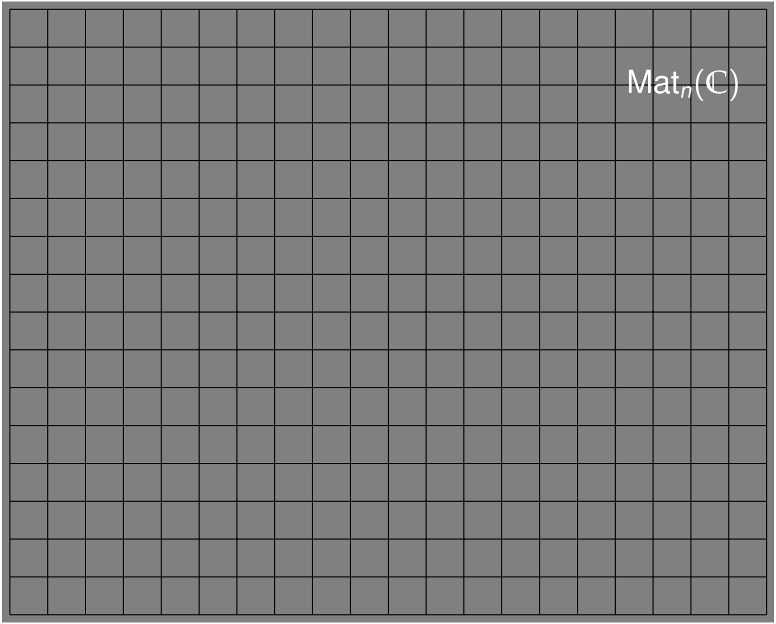
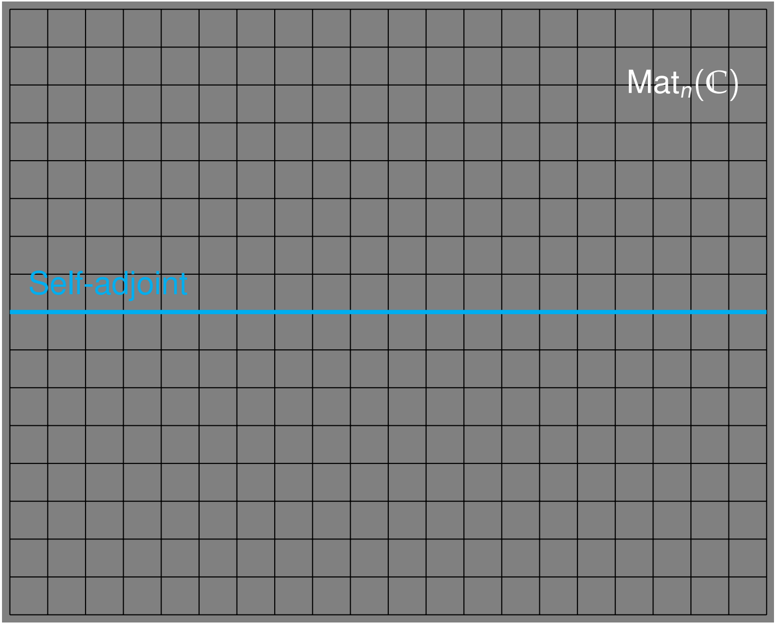
Spectral Theory

1. Quite dull in finite dimensions
2. Extends (non-trivial!) to infinite dimensions
3. Really important in infinite dimensions!
4. Gives a way of understanding infinite dimensional operators

Different Types of Operator



Different Types of Operator



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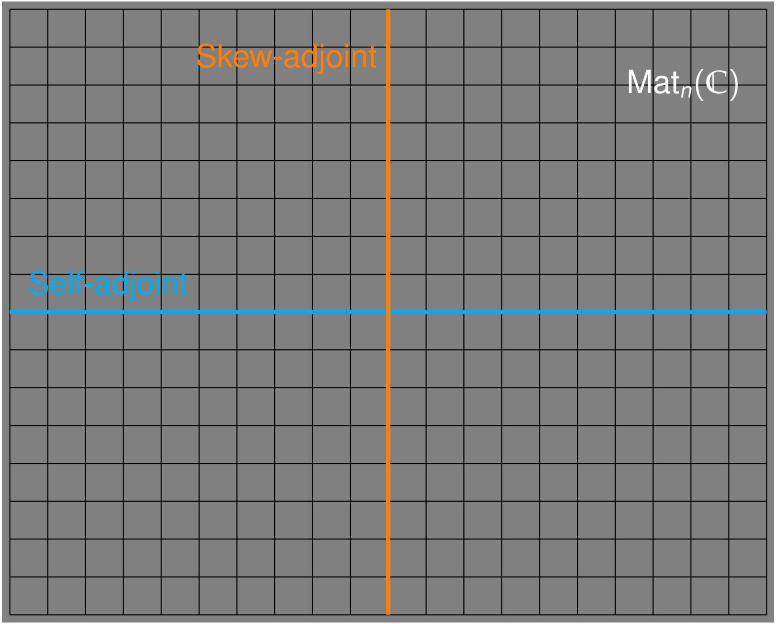

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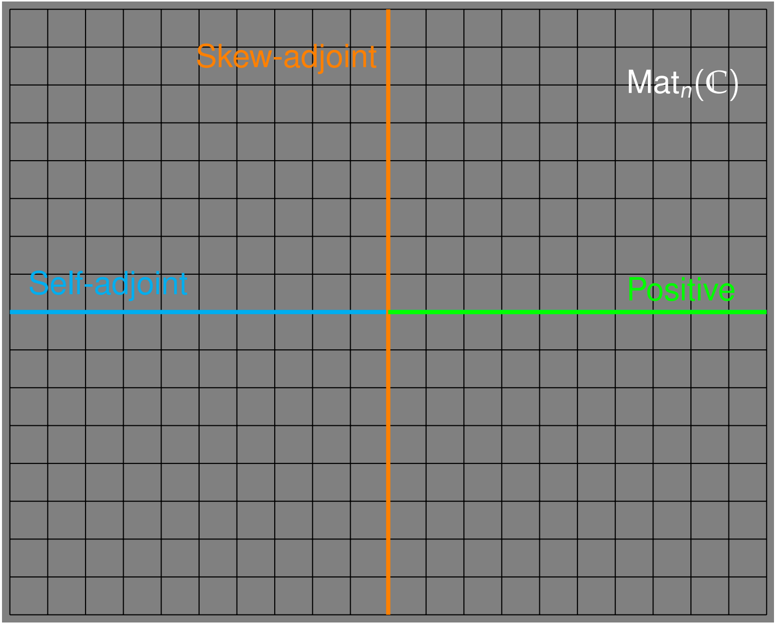

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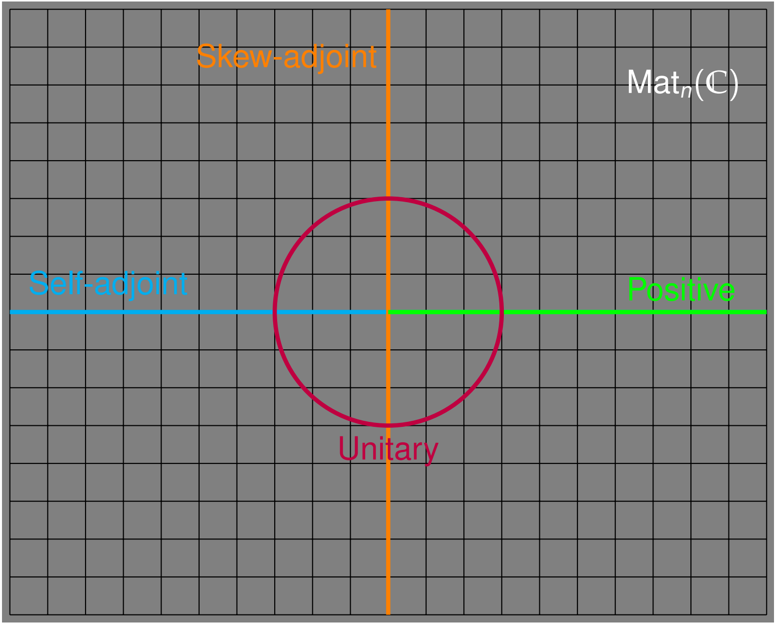

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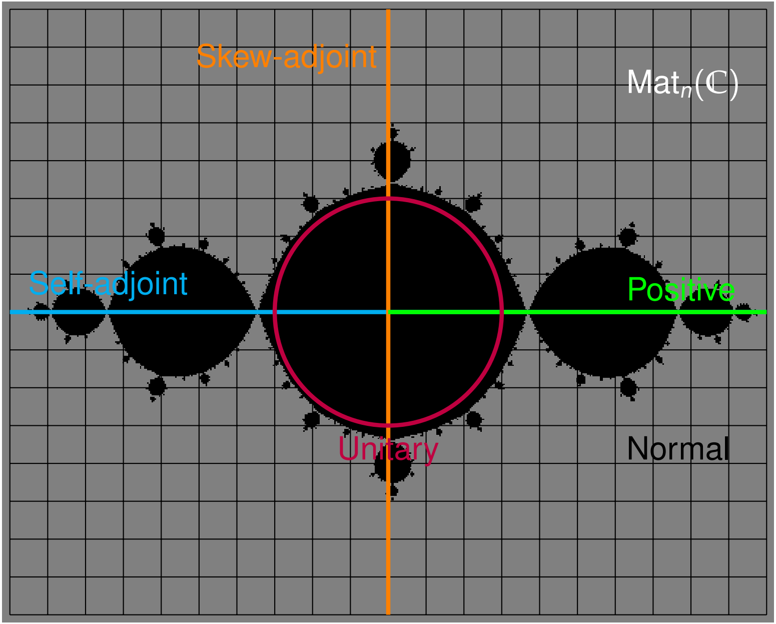

Summary

- Adjoints allow detailed study
- Nice behaviour under adjoint implies nice behaviour wrt inner product
- Normal $\iff$ orthogonally diagonalisable

Summary of Summaries

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Warning

This is not a “revision list” for the exam.

Table of Categories

Category	Objects	Morphisms
MSp	(M, d)	continuous, Lipschitz, pointwise continuous
VSp	$(V, +, 0, \lambda)$	linear
NVSp	$(V, \|\cdot\|)$	continuous + linear
BSp	Banach space	continuous + linear
IPSp	$(V, \langle\cdot,\cdot\rangle)$	continuous + linear, isometry
HSp	Hilbert space	continuous + linear, isometry

Use of Categories

MSp Home of sequences, place for convergence

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Use of Categories

MSp	Home of sequences, place for convergence
VSp	Linear $\implies$ decomposable
NVSp	Combine linearity and convergence
BSp	NVSp + nice limit behaviour
IPSp	Home of geometry
HSp	All of the above!

Metric Spaces

Purpose

Place to discuss convergence

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Examples

$(R, |\cdot|)$, $(S^n, \|p - q\|_2)$, $(\mathbb{N}_0, |1/p - 1/q|)$

Metric Spaces

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Key Technologies

- Sequences — convergent and Cauchy
- Neighbourhoods

Metric Spaces

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Key Technologies

- Sequences — convergent and Cauchy
- Neighbourhoods

Key Theorem

Banach's Fixed Point Theorem (Jacobi/Picard)

Vector Spaces

Purpose

Linearity means splits nicely

Vector Spaces

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Examples

$\mathbb{C}^n$, Poly_k

Vector Spaces

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Key Technologies

- Linear Transformations — matrices
- Factorisations

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Key Meta Theorem

Vector Spaces

Purpose

Linearity means splits nicely

Examples

$\mathbb{C}^n$, Poly_k

Key Technologies

- Linear Transformations — matrices
- Factorisations

Key Meta Theorem

Matrices factor into nice pieces.

Normed Vector Spaces/Banach Spaces

Purpose

Combines convergence with linearity

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Examples

ℓ^0, ℓ^1, ℓ^2, ℓ^∞, c_0, $C([0,1],\mathbb{C})$, $L^1(0,1)$, $L^2(0,1)$

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Key Technologies

- Sequences + Linear Transformations
- Series

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Key Technologies

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Key Theorem

... not really had one ...

Inner Product Spaces/Hilbert Spaces

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Combines linear, convergence, and geometry

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Key Technologies

- Orthogonality
- Duality

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Key Technologies

- Orthogonality
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Key Theorems

- Closest Point
- Riesz Representation

The Final Diagram

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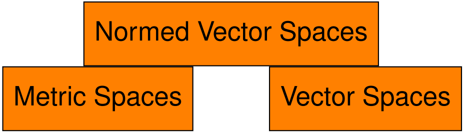

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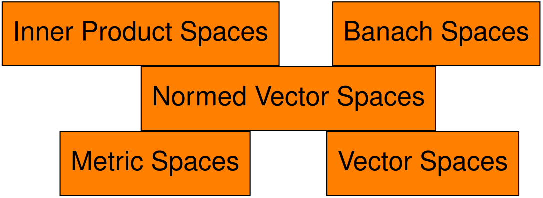

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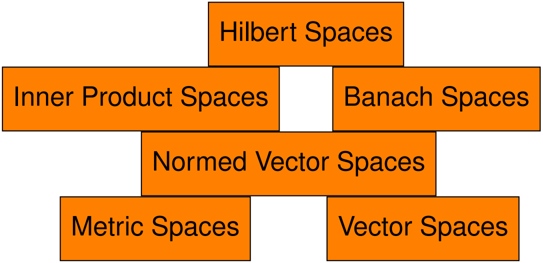
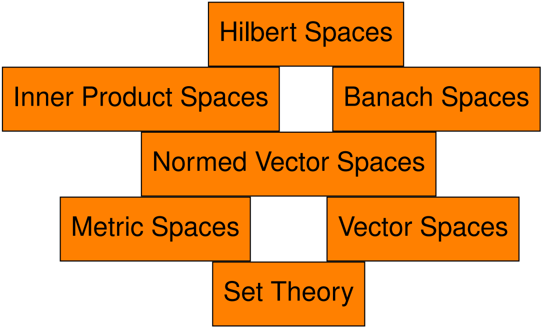


The Final Diagram



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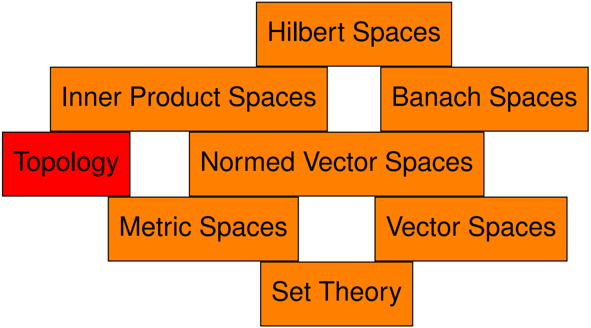

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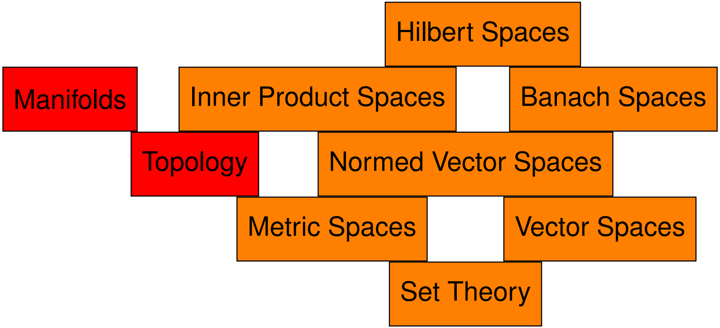

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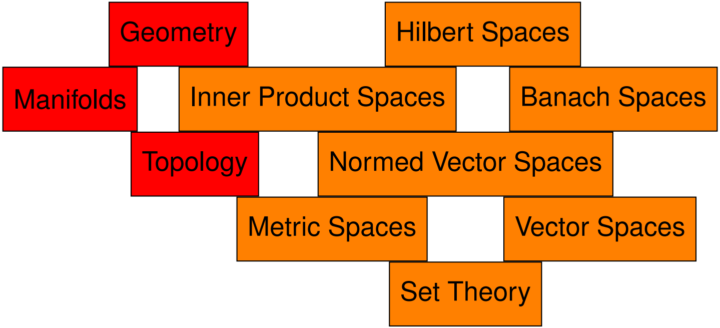

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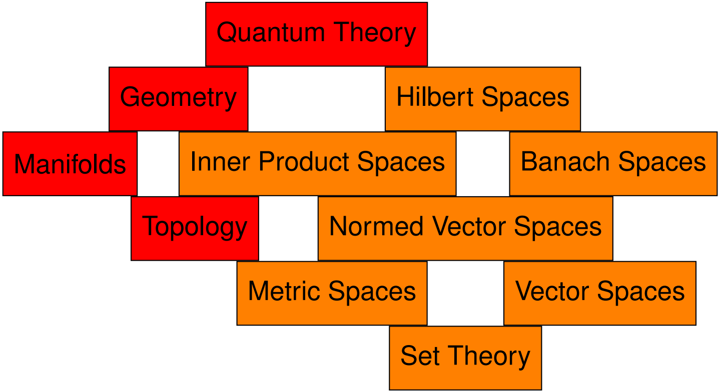

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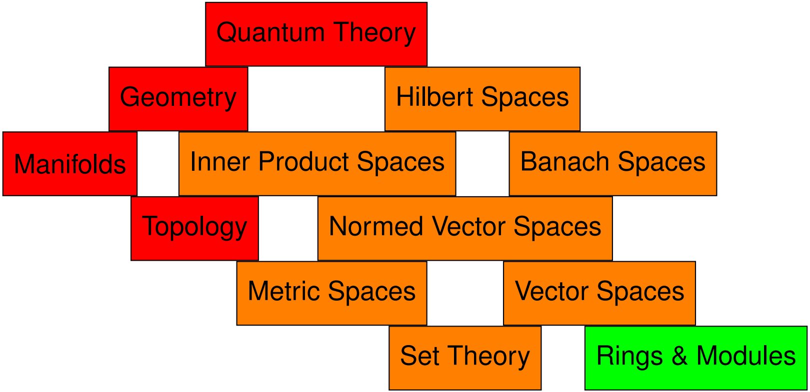
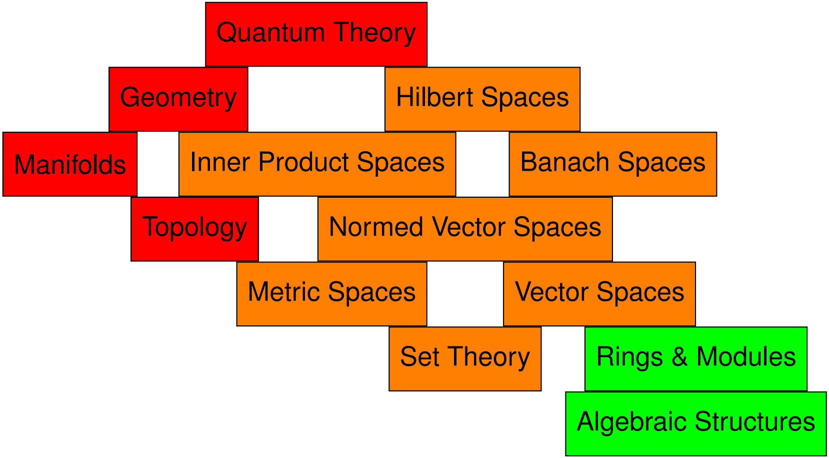


The Final Diagram



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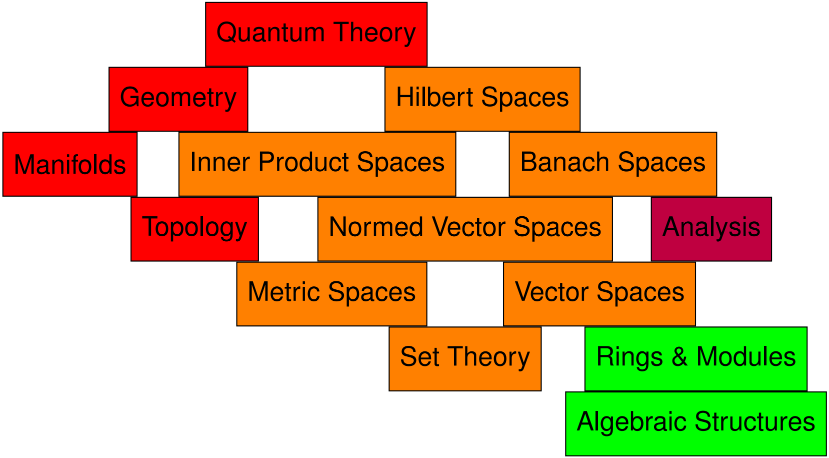

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