TMA4145 Linear Methods

Andrew Stacey

Norges Teknisk-Naturvitenskapelige Universitet
Trondheim

Autumn 2009

Lecture 28: Adroit Adjunctions Again And Adieu

Andrew Stacey

Norges Teknisk-Naturvitenskapelige Universitet
Trondheim

19th November 2009

Recap

Definition

H Hilbert space
$T \colon H \to H$ continuous linear.
*The **adjoint** of T is*
$T^\star \colon H \to H$
satisfying

$$\langle u, T^\star v \rangle = \langle Tu, v \rangle$$

self-adjoint:	$T = T^\star$
skew-adjoint:	$T = -T^\star$
unitary:	$U^{-1} = U^\star$
normal:	$TT^\star = T^\star T$

Key Point

- Make the most of coincidences!

Examples

1. $S\colon \ell^2 \to \ell^2$ shift operator:
   $S(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots)$

$$\begin{aligned}\langle S^\star(y_n), (x_n)\rangle &= \langle (y_n), S(x_n)\rangle \\ &= \langle (y_n), (x_{n-1})\rangle \\ &= \sum_{n\ge 2} y_n \overline{x_{n-1}} \\ &= \sum_{n\ge 1} y_{n+1}\overline{x_n} \\ &= \langle (y_{n+1}), (x_n)\rangle\end{aligned}$$

So $S^\star(y_1, y_2, y_3, \dots) = (y_2, y_3, \dots)$

Shift Operator

Then

$$\begin{aligned}SS^\star(x_1, x_2, x_3, \dots) &= (0, x_2, x_3, \dots) \\ S^\star S(x_1, x_2, x_3, \dots) &= (x_1, x_2, x_3, \dots)\end{aligned}$$

So not normal.

Example: Diagonal Operator

2. $D\colon \ell^2 \to \ell^2$: $D(x_1, x_2, x_3, \dots) = (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, \dots)$

$$\begin{aligned}\langle D^\star(y_n), (x_n)\rangle &= \langle (y_n), D(x_n)\rangle \\ &= \langle (y_n), (\lambda_n x_n)\rangle \\ &= \sum_{n\ge 1} y_n \overline{\lambda_n x_{n-1}} \\ &= \sum_{n\ge 1} \overline{\lambda_n} y_n \overline{x_n} \\ &= \langle (\overline{\lambda_n} y_n), (x_n)\rangle\end{aligned}$$

So $D^\star(x_1, x_2, x_3, \dots) = (\overline{\lambda_1} x_1, \overline{\lambda_2} x_2, \overline{\lambda_3} x_3, \dots)$

Example: Diagonal Operator

Then

$$\begin{aligned}DD^\star(x_1, x_2, x_3, \dots) &= (|\lambda_1|^2 x_1, |\lambda_2|^2 x_2, |\lambda_3|^2 x_3, \dots) \\ D^\star D(x_1, x_2, x_3, \dots) &= (|\lambda_1|^2 x_1, |\lambda_2|^2 x_2, |\lambda_3|^2 x_3, \dots)\end{aligned}$$

So

Self-adjoint	$\iff$	λ_n	$\in$	$\mathbb{R}$
Skew-adjoint	$\iff$	λ_n	$\in$	$i\mathbb{R}$
Unitary	$\iff$	λ_n	$\in$	S^1
Always normal				

Example: Matrix Operator

3. $A \in \mathrm{Mat}_n(\mathbb{C})$

$$\begin{aligned} \langle A^\star \vec{x}, \vec{y}\rangle &= \langle \vec{x}, A\vec{y}\rangle \\ &= \overline{\vec{x}}^\top A \vec{y} \\ &= \overline{\overline{A}^\top \vec{x}}^\top \vec{y} \\ &= \langle \overline{A}^\top \vec{x}, \vec{y}\rangle \end{aligned}$$

So $A^\star = \overline{A}^\top$

Example: Multiplication

4. $f \in L^2(0,1)$, $M_f \colon L^2(0,1) \to L^2(0,1)$ by $M_f g = fg$.

$$\begin{aligned} \langle M_f^\star g, h\rangle &= \langle g, M_f h\rangle \\ &= \int_0^1 g(t)\overline{(M_f h)(t)}dt \\ &= \int_0^1 g(t)\overline{f(t)h(t)}dt \\ &= \int_0^1 \overline{f(t)}g(t)\overline{h(t)}dt \\ &= \int_0^1 (M_{\bar{f}}\, g)(t)\overline{h(t)}dt \\ &= \langle M_{\bar{f}}\, g, h\rangle \end{aligned}$$

So $M_f{}^\star = M_{\bar{f}}$

Example: Multiplication

Then

$$\begin{aligned} M_f M_f{}^\star &= M_{|f|^2} \\ M_f{}^\star M_f &= M_{|f|^2} \end{aligned}$$

So

Self-adjoint	$\iff$	$f(t)$	$\in$	$\mathbb{R}$ all t
Skew-adjoint	$\iff$	$f(t)$	$\in$	$i\mathbb{R}$ all t
Unitary	$\iff$	$f(t)$	$\in$	S^1 all t
Always normal				

Note similarity to diagonal operators! Not a coincidence.

What's Normal Anyway?

- Diagonal $\Longrightarrow$ Normal
- U Diagonal $U^\star \Longrightarrow$ Normal

Theorem (Spectral Theorem in Finite Dimensions)

*$A \in \mathrm{Mat}_n(\mathbb{C})$ is **normal***

$\iff$

*it is **orthogonally diagonalisable***

Remark: Real version was $A = A^\top$

Proof: Very Easy Part

- A orthogonally diagonalisable

$\implies U \in \mathrm{Mat}_n(\mathbb{C})$ isometry so that $U^{-1}AU = D$ diagonal

- U isometry $\implies U^{-1} = U^\star$
- So $A = UDU^\star$
- And $A^\star = (UDU^\star)^\star = (U^\star)^\star D^\star U^\star = U\overline{D}U^\star$
- So

$$AA^\star = UDU^\star U\overline{D}U^\star = UD\overline{D}U^\star$$
$$A^\star A = U\overline{D}U^\star UDU^\star = U\overline{D}DU^\star$$

- But D diagonal so $D\overline{D} = \overline{D}D$
- Hence A normal.

Proof: Quite Easy Part

- A normal
- A has an eigenvector v, eigenvalue λ
- So $(A - \lambda I)v = 0$. Thus

$$\begin{aligned}
0 &= \left\|(A - \lambda I)v\right\|^2 \\
&= \langle (A - \lambda I)v, (A - \lambda I)v\rangle \\
&= \langle (A^\star - \overline{\lambda} I)(A - \lambda I)v, v\rangle \\
&= \langle (A - \lambda I)(A^\star - \overline{\lambda} I)v, v\rangle \\
&= \langle (A^\star - \overline{\lambda} I)v, (A^\star - \overline{\lambda} I)v\rangle \\
&= \left\|(A^\star - \overline{\lambda} I)v\right\|^2
\end{aligned}$$

So v eigenvector of $A^\star$, eigenvalue $\overline{\lambda}$.

Quite Easy Part

- Extend to orthonormal basis $\{v, u_2, \dots, u_n\}$
- Work out matrix of A:

$$\langle Av, v\rangle = \langle \lambda v, v\rangle = \lambda$$
$$\langle Av, u_j\rangle = \langle \lambda v, u_j\rangle = 0$$
$$\langle Au_j, v\rangle = \langle u_j, A^\star v\rangle = \langle u_j, \overline{\lambda} v\rangle = 0$$

- So matrix is:

$$\begin{bmatrix} \lambda & 0 \\ 0 & A_1 \end{bmatrix}$$

- A_1 also normal, so continue.

Example

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

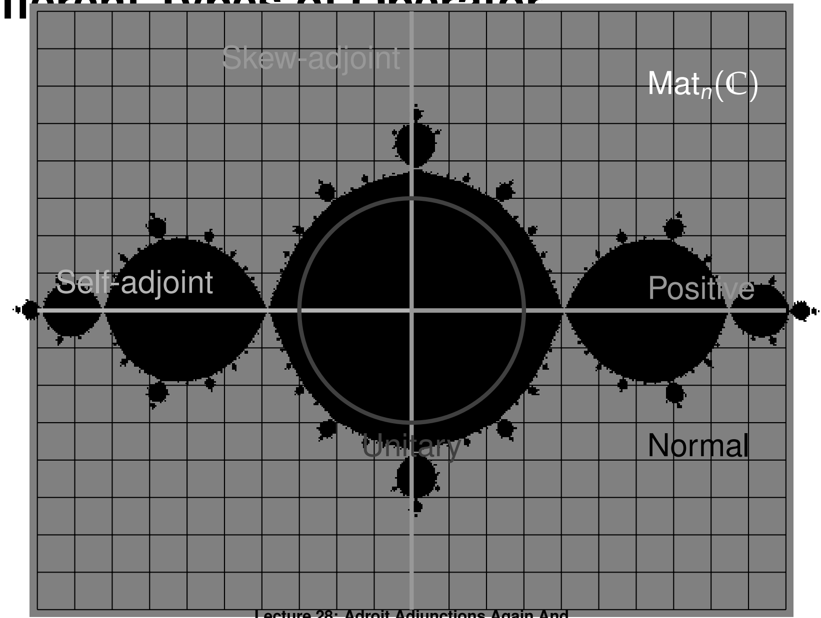
is unitary, hence normal.
So is orthogonally diagonalisable!

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$$

Spectral Theory

1. Quite dull in finite dimensions
2. Extends (non-trivial!) to infinite dimensions
3. Really important in infinite dimensions!
4. Gives a way of understanding infinite dimensional operators

Different Types of Operator



Summary

- Adjoints allow detailed study
- Nice behaviour under adjoint implies nice behaviour wrt inner product
- Normal $\iff$ orthogonally diagonalisable

Summary of Summaries

Warning

This is not a “revision list” for the exam.

Table of Categories

Category	Objects	Morphisms
MSp	(M, d)	continuous, Lipschitz, pointwise continuous
VSp	$(V, +, 0, \lambda)$	linear
NVSp	$(V, \|\cdot\|)$	continuous + linear
BSp	Banach space	continuous + linear
IPSp	$(V, \langle\cdot,\cdot\rangle)$	continuous + linear, isometry
HSp	Hilbert space	continuous + linear, isometry

Use of Categories

MSp Home of sequences, place for convergence
VSp Linear $\Longrightarrow$ decomposable
NVSp Combine linearity and convergence
BSp NVSp + nice limit behaviour
IPSp Home of geometry
HSp All of the above!

Metric Spaces

Purpose

Place to discuss convergence

Examples

$(R, |\cdot|)$, $(S^n, \|p - q\|_2)$, $(\mathbb{N}_0, |1/p - 1/q|)$

Key Technologies

- Sequences — convergent and Cauchy
- Neighbourhoods

Key Theorem

Banach's Fixed Point Theorem (Jacobi/Picard)

Vector Spaces

Purpose

Linearity means **splits nicely**

Examples

$\mathbb{C}^n$, Poly_k

Key Technologies

- Linear Transformations — matrices
- Factorisations

Key Meta Theorem

Matrices factor into **nice** pieces.

Normed Vector Spaces/Banach Spaces

Purpose

Combines **convergence** with **linearity**

Examples

ℓ^0, ℓ^1, ℓ^2, ℓ^∞, c_0, $C([0,1],\mathbb{C})$, $L^1(0,1)$, $L^2(0,1)$

Key Technologies

- Sequences + Linear Transformations
- Series

Key Theorem

. . . not really had one . . .

Inner Product Spaces/Hilbert Spaces

Purpose

Combines linear, convergence, and geometry

Examples

ℓ^2, $L^2(0,1)$

Key Technologies

- Orthogonality
- Duality

Key Theorems

- Closest Point
- Riesz Representation

The Final Diagram