

TMA4145 Linear Methods

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Lecture 28: Adroit Adjunctions Again And Adieu

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Recap

Definition

H Hilbert space

$T: H \rightarrow H$ continuous linear.

The *adjoint* of T is

$$T^*: H \rightarrow H$$

satisfying

$$\langle u, T^*v \rangle = \langle Tu, v \rangle$$

self-adjoint: $T = T^*$

skew-adjoint: $T = -T^*$

unitary: $U^{-1} = U^*$

normal: $TT^* = T^*T$

Key Point

- ▶ Make the most of coincidences!

Examples

1. $S: \ell^2 \rightarrow \ell^2$ shift operator:

$$S(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots)$$

$$\begin{aligned}\langle S^*(y_n), (x_n) \rangle &= \langle (y_n), S(x_n) \rangle \\ &= \langle (y_n), (x_{n-1}) \rangle \\ &= \sum_{n \geq 2} y_n \overline{x_{n-1}} \\ &= \sum_{n \geq 1} y_{n+1} \overline{x_n} \\ &= \langle (y_{n+1}), (x_n) \rangle\end{aligned}$$

$$\text{So } S^*(y_1, y_2, y_3, \dots) = (y_2, y_3, \dots)$$

Shift Operator

Then

$$SS^*(x_1, x_2, x_3, \dots) = (0, x_2, x_3, \dots)$$

$$S^*S(x_1, x_2, x_3, \dots) = (x_1, x_2, x_3, \dots)$$

So not normal.

Example: Diagonal Operator

$$2. D: \ell^2 \rightarrow \ell^2: D(x_1, x_2, x_3, \dots) = (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, \dots)$$

$$\begin{aligned}\langle D^*(y_n), (x_n) \rangle &= \langle (y_n), D(x_n) \rangle \\ &= \langle (y_n), (\lambda_n x_n) \rangle \\ &= \sum_{n \geq 1} y_n \overline{\lambda_n x_{n-1}} \\ &= \sum_{n \geq 1} \overline{\lambda_n y_n} \overline{x_n} \\ &= \langle (\overline{\lambda_n y_n}), (x_n) \rangle\end{aligned}$$

$$\text{So } D^*(x_1, x_2, x_3, \dots) = (\overline{\lambda_1 x_1}, \overline{\lambda_2 x_2}, \overline{\lambda_3 x_3}, \dots)$$

Example: Diagonal Operator

Then

$$DD^*(x_1, x_2, x_3, \dots) = (|\lambda_1|^2 x_1, |\lambda_2|^2 x_2, |\lambda_3|^2 x_3, \dots)$$

$$D^*D(x_1, x_2, x_3, \dots) = (|\lambda_1|^2 x_1, |\lambda_2|^2 x_2, |\lambda_3|^2 x_3, \dots)$$

So

Self-adjoint $\iff \lambda_n \in \mathbb{R}$

Skew-adjoint $\iff \lambda_n \in i\mathbb{R}$

Unitary $\iff \lambda_n \in \mathbb{S}^1$

Always normal

Example: Matrix Operator

3. $A \in \text{Mat}_n(\mathbb{C})$

$$\begin{aligned}\langle A^* \vec{x}, \vec{y} \rangle &= \langle \vec{x}, A \vec{y} \rangle \\ &= \vec{x}^T A \vec{y} \\ &= \overline{A^T \vec{x}^T} \vec{y} \\ &= \langle \overline{A^T} \vec{x}, \vec{y} \rangle\end{aligned}$$

So $A^* = \overline{A^T}$

Example: Multiplication

4. $f \in L^2(0, 1)$, $M_f: L^2(0, 1) \rightarrow L^2(0, 1)$ by $M_f g = fg$.

$$\begin{aligned}\langle M_f^* g, h \rangle &= \langle g, M_f h \rangle \\ &= \int_0^1 g(t) \overline{(M_f h)(t)} dt \\ &= \int_0^1 g(t) \overline{f(t)h(t)} dt \\ &= \int_0^1 \overline{f(t)} g(t) \overline{h(t)} dt \\ &= \int_0^1 (M_{\bar{f}} g)(t) \overline{h(t)} dt \\ &= \langle M_{\bar{f}} g, h \rangle\end{aligned}$$

So $M_f^* = M_{\bar{f}}$

Example: Multiplication

Then

$$M_f M_f^* = M_{|f|^2}$$

$$M_f^* M_f = M_{|f|^2}$$

So

Self-adjoint $\iff f(t) \in \mathbb{R}$ all t

Skew-adjoint $\iff f(t) \in i\mathbb{R}$ all t

Unitary $\iff f(t) \in S^1$ all t

Always normal

Note similarity to diagonal operators! Not a coincidence.

What's Normal Anyway?

- ▶ Diagonal \implies Normal
- ▶ U Diagonal U^* \implies Normal

Theorem (Spectral Theorem in Finite Dimensions)

$A \in \text{Mat}_n(\mathbb{C})$ is *normal*



it is *orthogonally diagonalisable*

Remark: Real version was $A = A^T$

Proof: Very Easy Part

- ▶ A orthogonally diagonalisable
- $\implies U \in \text{Mat}_n(\mathbb{C})$ isometry so that $U^{-1}AU = D$ diagonal
- ▶ U isometry $\implies U^{-1} = U^*$
- ▶ So $A = UDU^*$
- ▶ And $A^* = (UDU^*)^* = (U^*)^*D^*U^* = \overline{U}DU^*$
- ▶ So

$$AA^* = UDU^*\overline{U}DU^* = U\overline{D}DU^*$$

$$A^*A = \overline{U}DU^*UDU^* = \overline{U}DDU^*$$

- ▶ But D diagonal so $D\overline{D} = \overline{D}D$
- ▶ Hence A normal.

Proof: Quite Easy Part

- ▶ A normal
- ▶ A has an eigenvector v , eigenvalue λ
- ▶ So $(A - \lambda I)v = 0$. Thus

$$\begin{aligned}0 &= \|(A - \lambda I)v\|^2 \\&= \langle (A - \lambda I)v, (A - \lambda I)v \rangle \\&= \langle (A^* - \bar{\lambda}I)(A - \lambda I)v, v \rangle \\&= \langle (A - \lambda I)(A^* - \bar{\lambda}I)v, v \rangle \\&= \langle (A^* - \bar{\lambda}I)v, (A^* - \bar{\lambda}I)v \rangle \\&= \|(A^* - \bar{\lambda}I)v\|^2\end{aligned}$$

So v eigenvector of A^* , eigenvalue $\bar{\lambda}$.

Quite Easy Part

- ▶ Extend to orthonormal basis $\{v, u_2, \dots, u_n\}$
- ▶ Work out matrix of A :

$$\langle Av, v \rangle = \langle \lambda v, v \rangle = \lambda$$

$$\langle Av, u_j \rangle = \langle \lambda v, u_j \rangle = 0$$

$$\langle Au_j, v \rangle = \langle u_j, A^* v \rangle = \langle u_j, \bar{\lambda} v \rangle = 0$$

- ▶ So matrix is:

$$\begin{bmatrix} \lambda & 0 \\ 0 & A_1 \end{bmatrix}$$

- ▶ A_1 also normal, so continue.

Example

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

is unitary, hence normal.

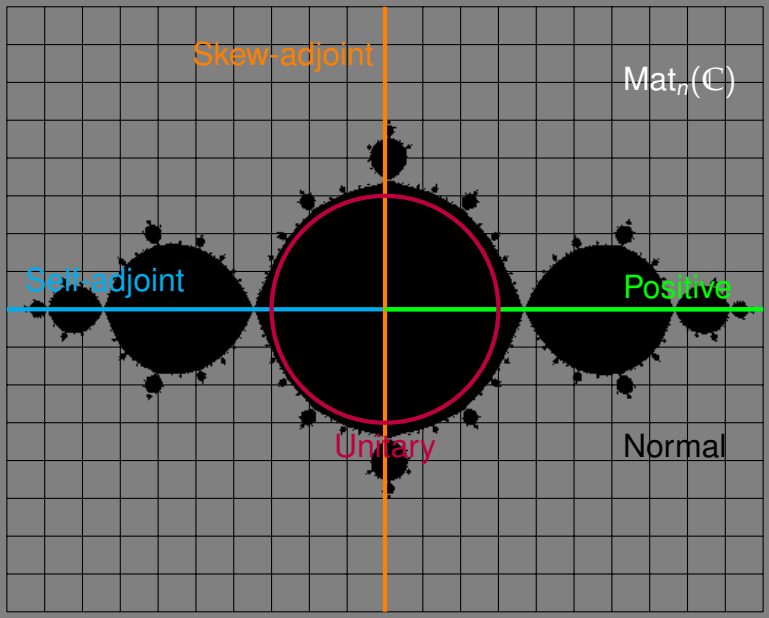
So is orthogonally diagonalisable!

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$$

Spectral Theory

1. Quite dull in finite dimensions
2. Extends (non-trivial!) to infinite dimensions
3. Really important in infinite dimensions!
4. Gives a way of understanding infinite dimensional operators

Different Types of Operator



Summary

- ▶ Adjoint allows detailed study
- ▶ Nice behaviour under adjoint implies nice behaviour wrt inner product
- ▶ Normal \iff orthogonally diagonalisable

Summary of Summaries

Warning

This is not a “revision list” for the exam.

Table of Categories

Category	Objects	Morphisms
MSp	(M, d)	continuous, Lipschitz, pointwise continuous
VSp	$(V, +, 0, \lambda)$	linear
NVSp	$(V, \ \cdot\)$	continuous + linear
BSp	Banach space	continuous + linear
IPSp	$(V, \langle \cdot, \cdot \rangle)$	continuous + linear, isometry
HSp	Hilbert space	continuous + linear, isometry

Use of Categories

- MSp** Home of sequences, place for convergence
- VSp** Linear \implies decomposable
- NVSp** Combine linearity and convergence
- BSp** NVSp + nice limit behaviour
- IPSp** Home of geometry
- HSp** All of the above!

Metric Spaces

Purpose

Place to discuss convergence

Examples

$(\mathbb{R}, |\cdot|)$, $(S^n, \|p - q\|_2)$, $(\mathbb{N}_0, |1/p - 1/q|)$

Key Technologies

- ▶ Sequences — convergent and Cauchy
- ▶ Neighbourhoods

Key Theorem

Banach's Fixed Point Theorem (Jacobi/Picard)

Vector Spaces

Purpose

Linearity means **splits nicely**

Examples

\mathbb{C}^n , Poly_k

Key Technologies

- ▶ Linear Transformations — matrices
- ▶ Factorisations

Key **Meta** Theorem

Matrices factor into **nice** pieces.

Normed Vector Spaces/Banach Spaces

Purpose

Combines **convergence** with **linearity**

Examples

$\ell^0, \ell^1, \ell^2, \ell^\infty, c_0, C([0, 1], \mathbb{C}), L^1(0, 1), L^2(0, 1)$

Key Technologies

- ▶ Sequences + Linear Transformations
- ▶ Series

Key Theorem

... not really had one ...

Inner Product Spaces/Hilbert Spaces

Purpose

Combines linear, convergence, and geometry

Examples

ℓ^2 , $L^2(0, 1)$

Key Technologies

- ▶ Orthogonality
- ▶ Duality

Key Theorems

- ▶ Closest Point
- ▶ Riesz Representation

The Final Diagram

