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English version

TMA4145 Linear Methods: Final Exam

Thursday 17th December 2009
Time: 09:00–13:00
Examination Aids: D
No written and handwritten examination support materials are permitted.
Calculator: Citizen SR-270X or Hewlett Packard HP30S

Problem 1.

Answer any **four** of the following.

i. Give the definition of a metric (function) on a set.
ii. Give the definition of a convergent sequence in a metric space.
iii. State Banach's Fixed Point Theorem.
iv. State the Spectral Theorem in finite dimensions.
v. Define a neighbourhood of a point in a metric space.
vi. State the Cauchy–Schwarz inequality.
vii. In the QR–factorisation of a matrix, what properties do Q and R have?

(8 points)

Problem 2.

Let (M, d) be a metric space.

1. Let (x_n) be a sequence in M converging to a point, say $x \in M$. Show that there is a subsequence (x_{n_k}) of (x_n) such that $d(x, x_{n_k}) < \frac{1}{k}$ for all $k \in \mathbb{N}$. *(2 points)*

   Solution:

   As $(x_n) \to x$, for each $k \in \mathbb{N}$ there is some $N_k \in \mathbb{N}$ such that whenever $n \geq N_k$ then $d(x, x_n) < \frac{1}{k}$. We now choose recursively $n_k \in \mathbb{N}$ such that $n_k > n_{k-1}$ and $n_k \geq N_k$. Then (x_{n_k}) is a subsequence of (x_n) as $n_k > n_{k-1}$ for all k, and $d(x, x_{n_k}) < \frac{1}{k}$ as $n_k \geq N_k$. Hence this is the required subsequence.

2. As before, let (x_n) be a sequence in M converging to a point, say $x \in M$. For each n, let $(x_n^m)_{m \in \mathbb{N}}$ be a sequence in M converging (as $m \to \infty$) to x_n. By applying part 1 to all of these sequences, or otherwise, prove that there is a sequence (y_k) such that

   (a) $(y_k) \to x$
   (b) for each $k \in \mathbb{N}$, there exist $n_k, m_k \in \mathbb{N}$ such that $y_k = x_{n_k}^{m_k}$
   (c) the n_k and m_k can be chosen such that $n_k > n_{k-1}$ and $m_k > m_{k-1}$.

(3 points)

Solution:

Following the hint, we apply part 1 to the sequence (x_n), to produce (x_{n_k}), and to the sequences (x_n^m), to produce $(x_n^{m_k})$. Then consider the "diagonal" sequence $(x_{n_k}^{m_k})$. By construction, this satisfies the second two properties; thus we just need to show that it converges to x. For that, we observe that

$$d(x, x_{n_k}^{m_k}) \leq d(x, x_{n_k}) + d(x_{n_k}, x_{n_k}^{m_k}) < \frac{1}{k} + \frac{1}{k} = \frac{2}{k}$$

and so $(x_{n_k}^{m_k}) \to x$.

3. Let $A \subseteq M$ be a subset. An accumulation point of A is a point $x \in M$ (not necessarily in A) for which there is a sequence in A converging to x. Show that if (x_n) is a sequence of accumulation points of A which converges in M then its limit is also an accumulation point of A. *(2 points)*

   Solution:

   We use the previous part. Let $x = \lim(x_n)$. As each x_n is an accumulation point of A, there is a sequence (x_n^m) converging to x_n with $x_n^m \in A$. By part 2, there is a sequence $(x_{n_k}^{m_k})$ drawn from the (x_n^m) which converges to x. As $x_{n_k}^{m_k} \in A$ for all k, this is a sequence in A converging to x and so x is an accumulation point of A.

4. In $(C([0,1], \mathbb{C}), \|\cdot\|_\infty)$ the following can be shown:

   (a) Every continuous function is the limit of a sequence of piecewise linear functions,

   (b) Every piecewise linear function is the limit of a sequence of polynomials.

   Explain why, for a continuous function $f : [0,1] \to \mathbb{C}$ and $\epsilon > 0$, there is a polynomial p such that $|f(t) - p(t)| < \epsilon$ for all $t \in [0,1]$. *(1 point)*

   Solution:

   By the above, there is a sequence of polynomials, say (p_n), converging to f in $C([0,1], \mathbb{C})$. Thus for $\epsilon > 0$, there is some n such that $\left\|f - p_n\right\|_\infty < \epsilon$. Writing out the definition of $\|\cdot\|_\infty$ now gives the result.

Problem 3.

A matrix A has QR–factorisation:

$$\begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 8 & -6 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

1. Explain why $A^\top A = R^\top R$ and compute this matrix. *(2 points)*

   Solution:

   Since $A = QR$ we see that $A^\top A = R^\top Q^\top QR$. As Q is orthogonal, $Q^\top Q = I$, whence $A^\top A = R^\top R$ which is

$$\begin{bmatrix} 64 & -48 & 0 \\ -48 & 64 & 0 \\ 0 & 0 & 25 \end{bmatrix}$$

2. Find an orthonormal basis of $\mathbb{R}^3$ of eigenvectors of $R^\top R$. *(3 points)*

   Solution:

   The most obvious eigenvector is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ with eigenvalue 25. We note that R was obviously not injective, so $R^\top R$ will not be injective and so will have an eigenvalue 0. Thus we look for the null space of $R^\top R$, which is $\begin{bmatrix} .6 \\ .8 \\ 0 \end{bmatrix}$. Finally, the third vector must be orthogonal to these two and so must be $\begin{bmatrix} .8 \\ -.6 \\ 0 \end{bmatrix}$ (or minus that). This has eigenvalue 100.

3. Find the singular value decomposition of R and hence find the singular value decomposition of A. *(3 points)*

Solution:

We apply R to the basis given in the previous part and obtain

$$\left\{\begin{bmatrix}10\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\5\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\0\end{bmatrix}\right\}$$

Thus we can take the standard basis for our basis of $\mathbb{R}^4$ and so write

$$R = \begin{bmatrix}10 & 0 & 0\\0 & 5 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}.8 & -.6 & 0\\0 & 0 & 1\\.6 & .8 & 0\end{bmatrix}$$

Since $A = QR$, the singular value decomposition for A is thus

$$\begin{bmatrix}1/2 & 1/2 & 1/2 & 1/2\\1/2 & -1/2 & 1/2 & -1/2\\1/2 & 1/2 & -1/2 & -1/2\\1/2 & -1/2 & -1/2 & 1/2\end{bmatrix}\begin{bmatrix}10 & 0 & 0\\0 & 5 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}.8 & -.6 & 0\\0 & 0 & 1\\.6 & .8 & 0\end{bmatrix}$$

Problem 4.

For $k \in \mathbb{N}$, let Poly_k be the vector space of polynomials with complex coefficients of degree at most k. Define a function $\mathrm{Poly}_3 \times \mathrm{Poly}_3 \to \mathbb{C}$ by

$$\langle p, q\rangle = \sum_{j=0}^{3} p(j)\overline{q(j)}$$

1. Prove that this is an inner product on Poly_3. *(3 points)*

Solution:

(a) Linearity:

$$\begin{aligned}\langle p_1 + \lambda p_2, q\rangle &= \sum_{j=0}^{3}(p_1(j) + \lambda p_2(j))\overline{q(j)}\\&= \sum_{j=0}^{3}\left(p_1(j)\overline{q(j)} + \lambda p_2(j)\overline{q(j)}\right)\\&= \sum_{j=0}^{3} p_1(j)\overline{q(j)} + \lambda\sum_{j=0}^{3} p_2(j)\overline{q(j)}\\&= \langle p_1, q\rangle + \lambda\langle p_2, q\rangle\end{aligned}$$

(b) Conjugate symmetry:

$$\begin{aligned}\langle p, q\rangle &= \sum_{j=0}^{3} p(j)\overline{q(j)}\\&= \sum_{j=0}^{3}\overline{q(j)\overline{p(j)}}\\&= \overline{\sum_{j=0}^{3} q(j)\overline{p(j)}}\\&= \overline{\langle q, p\rangle}\end{aligned}$$

(c) Non-degeneracy: suppose that $\langle p,p\rangle = 0$. Then $\sum_{j=0}^{3} |p(j)|^2 = 0$ so $p(j) = 0$ for $j \in \{0,1,2,3\}$. Hence $t(t-1)(t-2)(t-3)$ divides $p(t)$, but $p(t)$ has degree at most 3, whence $p(t) = 0$.

2. Apply the Gram–Schmidt algorithm to the family $\{1,t\}$ to find an orthonormal basis for Poly_1, regarded as a subspace of Poly_3 in the obvious way. *(2 points)*
Solution:
We start by working out the length of 1. This is

$$\langle 1,1\rangle = \sum_{j=0}^{3} 1 = 4$$

so we normalise to $1/2$.
Next, we orthogonalise t with respect to $1/2$. To do this, we compute:

$$\langle t, 1/2\rangle = \sum_{j=0}^{3} j1/2 = 1/2(0+1+2+3) = 3$$

Thus $\{1/2, t-3/2\}$ is orthogonal. We then work out the length of $t-3/2$. This is:

$$\langle t-3/2, t-3/2\rangle = \sum_{j=0}^{3} (j-3/2)^2 = 9/4 + 1/4 + 1/4 + 9/4 = 5$$

Hence our second polynomial is $1/\sqrt{5}(t-3/2)$.
Thus our orthonormal basis is:

$$\left\{1/2, 1/\sqrt{5}(t-3/2)\right\}$$

3. Let $p \in \mathrm{Poly}_3$ be a polynomial such that $p(0) = -3$, $p(1) = 1$, $p(2) = 3$, and $p(3) = 3$. Find the orthogonal projection of p on to Poly_1 (that is, find the closest point to p in the subspace Poly_1). *(2 points)*
Solution:
The closest point is given by

$$\begin{aligned}\langle p, 1/2\rangle 1/2 + \langle p, (t-3/2)/\sqrt{5}\rangle (t-3/2)/\sqrt{5} \\ = 1/4\langle p,1\rangle 1 + 1/5(\langle p,t\rangle - 3/2\langle p,1\rangle)(t-3/2)\end{aligned}$$

From the specification, we see that

$$\begin{aligned}\langle p,1\rangle &= p(0)+p(1)+p(2)+p(3) = 4 \\ \langle p,t\rangle &= p(1)+2p(2)+3p(3) = 16\end{aligned}$$

and so the closest point is

$$2t-2$$

4. Is $\deg p \leq 1$? *(1 point)*
Solution:
No. If it were, it would be $2t-2$ but this does not satisfy the specification as it takes on the value -2 at $t=0$.

Problem 5.

The Newton–Raphson method for finding roots of a differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ is to iterate the function

$$x \mapsto g(x) := x - \frac{f(x)}{f'(x)}.$$

We assume the following conditions on f:

i. f has continuous second derivative,

ii. $f'(x) \neq 0$ for all $x \in \mathbb{R}$,

iii. there is some $0 < \alpha < 1$ such that $|f(x)f''(x)| \leq \alpha |f'(x)|^2$ for all $x \in \mathbb{R}$.

1. Show that x^* is a fixed point of g if and only if $f(x^*) = 0$. *(1 point)*

   Solution:

   If $f(x^*) = 0$ then $g(x^*) = x^*$. Conversely, if $g(x^*) = x^*$ then $f(x^*)/f'(x^*) = 0$ so $f(x^*) = 0$.

2. Prove that g is a contraction. *(2 points)*

   You may find it useful to remember the Mean Value Theorem: that for a differentiable function $h\colon \mathbb{R} \to \mathbb{R}$ and $x < y \in \mathbb{R}$ then there is some $z \in (x, y)$ such that

   $$h(y) - h(x) = (y - x)h'(z).$$

   Solution:

   We need to compute $|g(y) - g(x)|$ in terms of $|y - x|$. Using the hint, we observe that it is sufficient to show that there is some constant, say β, which is less than 1 and such that $|g'(z)| \leq \beta$ for all $z \in \mathbb{R}$. Expanding out the definition of g and differentiating, we see that

   $$\begin{aligned} g'(z) &= 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{(f'(x))^2} \\ &= \frac{f(x)f''(x)}{(f'(x))^2}. \end{aligned}$$

   By assumption on f, this is strictly less than 1. Hence g is a contraction on $\mathbb{R}$.

3. Does this proceedure work with the polynomial $f(x) = x^k$ ($k \in \mathbb{N}$)? If not, which of the conditions fail? *(2 points)*

   Solution:

   It does not work because we don't have $f'(x) \neq 0$ for all x. However, we do have the other two conditions. In particular, looking at the third condition we see that on the left-hand side we have $k(k-1)|x|^{2k-2}$ and on the right-hand side we have $k^2|x|^{2k-2}$. Therefore we can take $\alpha = {}^{k-1}/_{k}$.

4. The proceedure does work for the polynomial $f(x) = x^3 + 3x + 1$ with $\alpha = {}^8/_9$. Using the estimate $|x_n - x^*| \leq \alpha^n |x_0 - x_1|$, estimate how many iterations would be needed to find x^* to within .01 starting with $x_0 = 0$. *(1 point)*

   Solution:

   We need to find n such that $\alpha^n |x_0 - x_1| \leq 0.01$. Starting with $x_0 = 0$, we get $x_1 = -{}^1/_3$, and so need $n \geq 29.771$. Thus $n = 30$ will do.

5. Do 5 iterations starting with $x_0 = 0$, recording them to an appropriate degree of accuracy. What do you observe? *(2 points)*

   Solution:

   The first five iterations (to 5 decimal places) are:

   $$-0.33333,\ -0.32222,\ -0.32219,\ -0.32219,\ -0.32219$$

   Clearly, this is converging much faster than we expected.