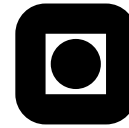


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English version

TMA4145 Linear Methods: Final Exam

Monday 6th December 2010

Time: 09:00–13:00

Examination Aids: D

No written or handwritten examination support materials are permitted.

Calculator: Citizen SR-270X or Hewlett Packard HP30S

Problem 1.

Answer any **four** of the following.

- i. Define a metric space (including the definition of a metric).
- ii. State the definition of a convergent sequence in a metric space.
- iii. Define a neighbourhood of a point in a metric space.
- iv. Define a linear transformation from one vector space to another.
- v. Give a definition of the dimension of a vector space.
- vi. Define an orthogonal family in an inner product space.
- vii. State the Riesz Representation Theorem.
- viii. State the Cauchy–Schwarz inequality.

Problem 2.

1. Let (M, d) be a metric space. Let (x_n) be a convergent sequence in (M, d) , with limit $x \in M$. Prove that (x_n) is a Cauchy sequence.

Solution:

Let $\epsilon > 0$. As (x_n) converges to x , there is some $n \in \mathbb{N}$ such that for $k \geq n$, $d(x_k, x) < \epsilon/2$. Then for $k, l \geq n$, the triangle inequality shows that $d(x_k, x_l) \leq d(x_k, x) + d(x, x_l)$ and hence $d(x_k, x_l) < \epsilon$. Thus (x_n) is Cauchy.

2. Recall that for $\theta, \phi \in \mathbb{R}$:

- (a) $\cos(\theta - \phi) - \cos(\theta + \phi) = 2 \sin(\theta) \sin(\phi)$
- (b) $|\sin(\theta)| \leq |\theta|$

Show that $\cos: [0, 1] \rightarrow [0, 1]$ is a contraction.

(Note: \cos and \sin are defined using *radians*.)

Solution:

First, we observe that if $x \in [0, 1]$ then $\cos(x) \in [0, 1]$ since $\cos(0) = 1$, $\cos(\pi/2) = 0$, and \cos is decreasing on the interval $[0, \pi]$.

Then for $x, y \in [0, 1]$, let us put $\theta = (x + y)/2$ and $\phi = (x - y)/2$ in the first given formula. Then $\theta + \phi = x$ and $\theta - \phi = y$ so

$$\cos(x) - \cos(y) = 2 \sin\left(\frac{x + y}{2}\right) \sin\left(\frac{x - y}{2}\right).$$

Taking absolute values, we have:

$$|\cos(x) - \cos(y)| = 2 \left| \sin\left(\frac{x + y}{2}\right) \sin\left(\frac{x - y}{2}\right) \right|.$$

Now using the second fact, we write this as an inequality:

$$|\cos(x) - \cos(y)| \leq 2 \left| \sin\left(\frac{x + y}{2}\right) \right| \left| \frac{x - y}{2} \right| = \left| \sin\left(\frac{x + y}{2}\right) \right| |x - y|.$$

Since \sin is increasing on $[0, 1]$, the largest possible value for $\sin\left(\frac{x + y}{2}\right)$ is $\sin(1) \approx 0.842 \leq 0.85$. Hence

$$|\cos(x) - \cos(y)| \leq 0.85 |x - y|.$$

3. Explain why there is some $x_0 \in [0, 1]$ with $x_0 = \cos(x_0)$ and describe a procedure to approximate it.

Solution:

The set $[0, 1]$ is a closed subset of \mathbb{R} . As \mathbb{R} is complete, $[0, 1]$ is therefore also complete. It is clearly not empty. Thus $\cos: [0, 1] \rightarrow [0, 1]$ is a contraction on a complete, non-empty metric space. Banach's fixed point theorem therefore applies and so we can conclude that there is a (unique) fixed point. That is, there is some $x_0 \in [0, 1]$ such that $x_0 = \cos(x_0)$.

Banach's fixed point theorem not only tells us that this point exists, it also tells us one way to approximate it. We start with any point in $[0, 1]$, say 0, and repeatedly apply \cos . That is to say, the sequence defined by $x_1 = 0$ and $x_n = \cos(x_{n-1})$ will converge to x_0 .

Problem 3.

For $k \in \mathbb{N}$, let Poly_k be the vector space of polynomials of degree at most k with real coefficients. Let $V := \{p(t) \in \text{Poly}_2 : p'(0) = 0\}$ (note the 2).

1. Explain why V is a vector space and find an isomorphism $\mathbb{R}^n \cong V$ for some $n \in \mathbb{N}$ (which you should determine).

Solution:

It is stated in the question that Poly_2 is a vector space and therefore we can use the subspace criterion to determine whether or not V is a subspace. Thus we simply need to check that V is closed under the vector space operations.

- (a) The zero vector in Poly_2 is the zero polynomial, $0(t)$, and for all $t \in \mathbb{R}$, $0'(t) = 0$ so $0(t) \in V$.

(b) For $p(t) \in V$ and $\lambda \in \mathbb{R}$ then $(\lambda p)'(t) = \lambda p'(t)$ so $(\lambda p)'(0) = \lambda p'(0) = 0$. Hence $\lambda p(t) \in V$.

(c) For $p(t), q(t) \in V$ then $(p+q)'(t) = p'(t)+q'(t)$ so $(p+q)'(0) = p'(0)+q'(0) = 0$. Hence $p(t) + q(t) \in V$.

To find an isomorphism $\mathbb{R}^2 \rightarrow V$ we examine the elements of V a little closer. A polynomial in Poly_2 has the form $x_1 + x_2t + x_3t^2$. Its derivative is $x_2 + 2x_3t$ which evaluates at 0 to x_2 . Therefore $x_1 + x_2t + x_3t^2$ is in V if and only if $x_2 = 0$; that is, if and only if it is of the form $x_1 + x_3t^2$. We can read off from this an isomorphism $\mathbb{R}^2 \rightarrow V$ given by $\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow x + yt^2$.

2. For $a, b \in \mathbb{R}$, define $T_{a,b}: V \rightarrow \mathbb{R}^2$ by $T_{a,b}(p(t)) = \begin{bmatrix} p(a) \\ p(b) \end{bmatrix}$. For which pairs of values (a, b) is $\ker T_{a,b} = \{0\}$?

Solution:

There are a couple of ways to do this.

One is to use the fact that a polynomial of degree at most 2 is completely specified by its values at three distinct points, together with an obvious fact about the shape of the polynomials which lie in V . This fact is that these polynomials are symmetric about the y -axis. Thus if $p(t) \in V$ then for all $t \in \mathbb{R}$, $p(-t) = p(t)$.

So if $a \neq b$ then $\{a, b\}$ provides two points. Using the fact that our polynomials are even, we potentially gain two more points: $\{-a, -b\}$. Thus we have four points unless there is some overlap between $\{a, b\}$ and $\{-a, -b\}$. To see what may occur if there is some overlap, suppose that $a = -a$. Then $a = 0$ and since we assumed that $b \neq a$, we must have that $b \neq -b$ and so $\{a, b, -b\}$ are three distinct points. Now suppose that $a = -b$. Then $b = -a$ and there are only two points. Thus if $b \neq a$ and $b \neq -a$ we have at least three distinct points. If $b = a$, we get at most two points, being $\{a, -a\}$, and possibly only one (if $a = 0$).

Hence $\ker T_{(a,b)} = \{0\}$ if and only if $b \notin \{a, -a\}$.

The other way to do this is to use the isomorphism $\mathbb{R}^2 \rightarrow V$ constructed in the previous part. By composition, we obtain a linear transformation $\mathbb{R}^2 \rightarrow \mathbb{R}^2$. As this is the composition of $T_{(a,b)}$ with an isomorphism, this new linear transformation will have trivial kernel (i.e., $\{0\}$) if and only if $T_{(a,b)}$ does. The matrix of this new linear transformation is:

$$\begin{bmatrix} 1 & a^2 \\ 1 & b^2 \end{bmatrix}$$

(different isomorphisms in part one will result in different matrices here, however the analysis will be the same). There are lots of ways to determine whether or not this is injective. Perhaps the simplest is simply by observation: it will be injective if and only if the two columns are linearly independent. The second is a multiple of the first if and only if $a^2 = b^2$. Equivalently, if and only if $b \in \{a, -a\}$.

3. For $a, b \in \mathbb{R}$, define $R_{a,b}: V \rightarrow \text{Poly}_1$ by sending $p(t) \in V$ to the polynomial $q(t) \in \text{Poly}_1$ with $q(0) = p(a)$ and $q(1) = p(b)$. For which pairs of values (a, b) is $R_{a,b}$ an isomorphism?

Solution:

We can write this linear transformation as the composition of $T_{(a,b)}$ from the previous part with the isomorphism $\mathbb{R}^2 \rightarrow \text{Poly}_1$ which sends $\begin{bmatrix} x \\ y \end{bmatrix}$ to the polynomial $q(t)$ with $q(0) = x$ and $q(1) = y$ (explicitly, $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto x + (y - x)t$). As the second factor in this composition is an isomorphism, the whole will be an isomorphism if and only if $T_{(a,b)}$ is an isomorphism. Since V and \mathbb{R}^2 are both 2-dimensional, $T_{(a,b)}$ is an isomorphism if and only if $\ker T_{(a,b)} = \{0\}$. Thus $R_{(a,b)}$ is an isomorphism if and only if $b \notin \{a, -a\}$.

Problem 4.

Define $W \subseteq \mathbb{R}^4$ as the subspace:

$$W := \left\{ \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \text{ with } \begin{cases} -4w + 4x + 3y + 5z = 0, \\ -6w + 6x + 3y + 9z = 0, \\ -2w + 2x + 4z = 0, \\ 2w - 2x - 3y - z = 0 \end{cases} \right\}$$

Find the closest point in W to the vector

$$\begin{bmatrix} 7 \\ -5 \\ -1 \\ -3 \end{bmatrix}.$$

Solution:

There are two basic methods to solve this: either find an orthonormal basis for W and use the projection formula or use the method of least-squares. Both begin in the same way.

The subspace W is described in the question as the null space of a matrix. We need to describe it as the image of a matrix, or find a basis for it. To do this, we run Gaussian Elimination on the matrix in the question. The resulting row reduced form is:

$$\begin{bmatrix} 1 & -1 & 0 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis for the null space is thus:

$$\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

and so W is the image of the matrix

$$\begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$$

At this point, we could orthonormalise the basis or use the method of leastsquares using the matrix. Swapping and then orthonormalising leads to the vectors

$$\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(the swap is to get a “nice” orthonormal basis). Then our closest point is:

$$\left\langle \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ -5 \\ -1 \\ -3 \end{bmatrix} \right\rangle \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \left\langle \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ -5 \\ -1 \\ -3 \end{bmatrix} \right\rangle \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2} 2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{4} 8 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 2 \end{bmatrix}$$

Proceeding via least-squares, we solve $A^T A x = A^T b$ where A is the matrix whose image is W and b is the vector in the question. This leads us to:

$$\begin{bmatrix} 6 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$$

The appropriate Gaussian Elimination leads to the solution $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$. Applying A yields the closest point as:

$$\begin{bmatrix} 3 \\ -1 \\ 2 \\ 2 \end{bmatrix}$$

as before.

Problem 5.

This question concerns $C([0, 1], \mathbb{C})$ with its standard inner product:

$$\langle f, g \rangle := \int_0^1 f(t) \overline{g(t)} dt.$$

For this question, we choose $a, b \in (0, 1)$ such that $a < b$.

1. Show that the linear function $\alpha: C([0, 1], \mathbb{C}) \rightarrow \mathbb{C}$ defined by

$$\alpha(f) := \int_0^1 f(t) dt$$

is continuous.

Solution:

This linear function can be written using the inner product. Let $\mathbb{1}$ be the constant function at 1, then

$$\alpha(f) = \langle f, \mathbb{1} \rangle.$$

Hence, by the Cauchy–Schwarz inequality,

$$|\alpha(f)| \leq \|f\|_2 \|\mathbb{1}\|_2$$

where $\|\cdot\|_2$ is the norm defined by the inner product. Since $\|\mathbb{1}\|_2 = (\int_0^1 1^2 dt)^{1/2} = 1$, this simplifies to $|\alpha(f)| \leq \|f\|_2$. Hence α is Lipschitz and thus continuous.

2. Define the linear function $\beta: C([0, 1], \mathbb{C}) \rightarrow \mathbb{C}$ by

$$\beta(f) := \int_a^b f(t)dt.$$

Show that β is continuous.

Solution:

We can't use quite the same trick as in the first part, but we can use the fact that α is continuous to deduce that β is also continuous by bounding β by α . We have the following chain of inequalities:

$$|\beta(f)| = \left| \int_a^b f(t)dt \right| \leq \int_a^b |f(t)|dt \leq \int_0^1 |f(t)|dt = \alpha(|f|) \leq \| |f| \|_2$$

Since $\| |f| \|_2 = \|f\|_2$, we therefore have that $|\beta(f)| \leq \|f\|_2$, whence β is continuous.

3. Explain why there is some $g \in L^2(0, 1)$ such that for all $f \in C([0, 1], \mathbb{C})$,

$$\langle f, g \rangle = \int_a^b f(t)dt.$$

Is g an element of $C([0, 1], \mathbb{C})$?

Solution:

As β is Lipschitz continuous, it extends to a linear functional on $L^2(0, 1)$, the Hilbert completion of $C([0, 1], \mathbb{C})$. By the Riesz Representation Theorem, there is therefore an element $g \in L^2(0, 1)$ such that for all $f \in L^2(0, 1)$

$$\beta(f) = \langle f, g \rangle.$$

In particular, this holds for $f \in C([0, 1], \mathbb{C})$.

The element g is not an element of $C([0, 1], \mathbb{C})$. To prove this carefully is quite complicated, but intuitively this is obvious as g "ought to be" the "function":

$$g(t) = \begin{cases} 1 & a \leq t \leq b \\ 0 & \text{elsewhere} \end{cases}$$

which is not continuous at a or b .