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English version

TMA4145 Linear Methods: Continuation Exam

Saturday 15th August 2009
Time: 09:00–13:00
Examination Aids: D
No written and handwritten examination support materials are permitted.
Calculator: Citizen SR-270X or Hewlett Packard HP30S

Problem 1.

Give the definitions of any **four** of the following terms.

i. A convergent sequence in a metric space
ii. The adjoint of a linear transformation between inner product spaces
iii. A neighbourhood of a point in a metric space
iv. The image (column space) of a linear transformation
v. A complete metric space
vi. A normal linear transformation
vii. A basis in a vector space

(8 points)

Problem 2.

a. Let $(X, \|\cdot\|)$ be a Banach space. Let $T\colon X \to X$ be a continuous linear transformation. Let $b \in X$ be a fixed vector.
Consider the map $X \to X$, $x \mapsto Tx + b$. Express as simply as you can the condition that is needed for this to be a contraction. *(3 points)*
Solution:
A map $g\colon M \to M$ on a metric space is a contraction if there is some $\alpha < 1$ such that $d(g(x), g(y)) \leq \alpha d(x, y)$ for all $x, y \in M$. In this case, we have $g(x) = Tx + b$ and $d(u, v) = \|u - v\|$. Substituting in, we obtain the condition that there is some $\alpha < 1$ such that

$$\left\|Tx + b - (Ty + b)\right\| \leq \alpha \left\|x - y\right\|$$

which simplifies to $\left\|T(x - y)\right\| \leq \alpha \left\|x - y\right\|$. We can simplify this further to $\|Tz\| \leq \alpha \|z\|$ by noting that if the former holds, then it holds for $y = 0$, and if the latter holds then it holds for $z = x - y$. Thus the condition is that there is some $\alpha < 1$ such that $\|Tz\| \leq \alpha \|z\|$ for all $z \in X$.

b. Recall that ℓ^∞ is the space of all *bounded* sequences of real numbers with the supremum norm,

$$\|(\xi_n)\|_\infty = \sup\{|\xi_n| : n \in \mathbb{N}\}$$

Let $I\colon \ell^\infty \to \ell^\infty$ be the map

$$I(\xi_n) = \left(\frac{1}{n}\xi_{n+1}\right).$$

For example,

$$I(1,1,1,1,1,\dots) = \left(0, 1, \frac{1}{2}, \frac{1}{3}, \dots\right).$$

Let $b \in \ell^\infty$ be the sequence $(1,0,0,\dots)$. Prove that the map $x \mapsto Ix + b$ is not a contraction on ℓ^∞ but that $x \mapsto I(Ix+b)+b$ is a contraction. What is the contraction constant?

(3 points)

Note: *There was a mistake in this question. The definition of I should be*

$$I(\xi_n) = \left(0, \xi_1, \frac{1}{2}\xi_2, \frac{1}{3}\xi_3, \dots\right)$$

The example was correct.

Solution:

The map $x \mapsto Ix + b$ is not a contraction since $I(e_1) = e_2$ and thus $\|Ie_1\| = 1 = \|e_1\|$. However, $x \mapsto I(Ix+b)+b)$ is a contraction. This is of the form $x \mapsto Tx + c$ where $T = I^2$ (and $c = Ib + b$). Written out, I^2 is

$$I^2(\xi_n) = \left(0, 0, \frac{1}{2}\xi_1, \frac{1}{6}\xi_2, \frac{1}{12}\xi_3, \dots\right)$$

from which we see that each coefficients is multiplied by $\frac{1}{k(k-1)}$ and thus $\left\|I^2(x)\right\| \le \frac{1}{2}\|x\|$, with equality when $x = e_1$. Hence $x \mapsto I(Ix+b)+b$ is a contraction with contraction constant $\frac{1}{2}$.

c. Carry out the first five iterations (of $x \mapsto Ix + b$) starting from $x_0 = 0$. Then carry out the first five iterations starting from an arbitrary sequence $x_0 = (\xi_1, \xi_2, \dots)$. What do you notice?

(2 points)

Solution:

The first five iterations with $x_0 = 0$ are:

$$\begin{gathered}
(1,0,0,0,0,0,\dots) \\
(1,1,0,0,0,0,\dots) \\
(1,1,\tfrac{1}{2},0,0,0,\dots) \\
(1,1,\tfrac{1}{2},\tfrac{1}{6},0,0,\dots) \\
(1,1,\tfrac{1}{2},\tfrac{1}{6},\tfrac{1}{24},0,\dots)
\end{gathered}$$

The first five iterations with $x_0 = (\xi_n)$ are:

$$\begin{gathered}
(1,\xi_1,\tfrac{1}{2}\xi_2,\tfrac{1}{3}\xi_3,\tfrac{1}{4}\xi_4,\tfrac{1}{5}\xi_5,\dots) \\
(1,1,\tfrac{1}{2}\xi_1,\tfrac{1}{6}\xi_2,\tfrac{1}{12}\xi_3,\tfrac{1}{20}\xi_4,\dots) \\
(1,1,\tfrac{1}{2},\tfrac{1}{6}\xi_1,\tfrac{1}{24}\xi_2,\tfrac{1}{60}\xi_3,\dots) \\
(1,1,\tfrac{1}{2},\tfrac{1}{6},\tfrac{1}{24}\xi_1,\tfrac{1}{120}\xi_2,\dots) \\
(1,1,\tfrac{1}{2},\tfrac{1}{6},\tfrac{1}{24},\tfrac{1}{120}\xi_1,\dots)
\end{gathered}$$

The iterations seem to get close very quickly, the dependence on the initial vector is very small.

Problem 3.

Let $(X, \|\cdot\|)$ be a normed vector space.

a. Let (a_n) be a sequence of points in X with the property that the series $\sum \|a_n\|$ converges in $\mathbb{R}$. Prove that the sequence (s_n) in X, defined by

$$s_n = \sum_{k=1}^{n} a_k$$

is Cauchy. *(3 points)*

Solution:

By the triangle inequality, we have

$$\|s_n\| \leq \sum_{k=1}^{n} \|a_k\| .$$

By assumption, $\sum \|a_n\|$ converges in $\mathbb{R}$ and thus its sequence of partial sums, say (t_n), converges. Hence this sequence of partial sums is Cauchy and so for $\epsilon > 0$ there is some N such that for $n, m \geq N$, $|t_n - t_m| < \epsilon$. Assuming, without loss of generality, that $n \geq m$, we have $t_n - t_m = \sum_{k=m+1}^{n} \|a_k\|$. Thus whenever $n \geq m \geq N$,

$$\|s_n - s_m\| = \left\| \sum_{k=m+1}^{n} a_k \right\| \leq \sum_{k=m+1}^{n} \|a_k\| = |t_n - t_m| < \epsilon$$

and hence (s_n) is Cauchy.

b. Prove that a normed vector space $(X, \|\cdot\|)$ is complete if and only if whenever (a_n) is a sequence in X with the property that $\sum \|a_n\|$ converges then the sequence of partial sums (s_n) with $s_n = \sum_{k=1}^{n} a_k$ converges in X.

You may assume that a Cauchy sequence (s_n) in X has a subsequence (s_{n_m}) with the property that $\left\| s_{n_{m+1}} - s_{n_m} \right\| < \frac{1}{2^m}$. *(3 points)*

Solution:

From above, if $(X, \|\cdot\|)$ is complete then as (s_n) is Cauchy, it converges.

To prove the converse, let (s_n) be a Cauchy sequence. We need to construct a series (a_n) such that $\sum \|a_n\|$ converges in $\mathbb{R}$. Unfortunately, $a_n = s_n - s_{n+1}$ may not work. Using the hint, though, we first choose a subsequence of (s_n), say (s_{n_m}), with the property that $\left\| s_{n_{m+1}} - s_{n_m} \right\| < \frac{1}{2^m}$. Then we define $a_k := s_{n_{k+1}} - s_{n_k}$. By construction, $\|a_k\| < \frac{1}{2^k}$ and thus $\sum \|a_k\|$ converges. Hence $(s_{n_m}$, which is the sequence of partial sums of $\sum a_k$, converges by assumption. But then (s_n) is a Cauchy sequence with a convergent subsequence and hence converges.

c. Recall that ℓ^0 is the space of all sequences in $\mathbb{R}$ that are eventually zero. Let $\|\cdot\|$ be a norm on ℓ^0. Prove that $(\ell^0, \|\cdot\|)$ is not a Banach space. *(2 points)*

Solution:

We shall find a series that ought to converge but doesn't. Let $\lambda_n = \|e_n\|$ and let $\mu_n = \frac{1}{2^n \lambda_n}$. Then consider the series $\sum \mu_n e_n$. The corresponding series in $\mathbb{R}$ is $\sum \mu_n \|e_n\| = \sum \frac{1}{2^n}$ which converges. However, it is easy to see that this series does not converge in ℓ^0 as its "limit" will be $(\mu_1, \mu_2, \mu_3, \ldots)$ which is not in ℓ^0.

Problem 4.

In this problem we equip $\mathbb{R}^3$ and $\mathbb{R}^4$ with their standard Euclidean norms:

$$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2} \qquad (n = 3 \text{ or } 4)$$

Let

$$A = \begin{bmatrix} 5 & 2 & 1 \\ -1 & -1 & 1 \\ 3 & 2 & -1 \\ -3 & -1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} -3 \\ -3 \\ 3 \\ 3 \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ 3 \\ 12 \\ -9 \end{bmatrix}$$

a. Find an $x \in \mathbb{R}^3$ such that $Ax = b$. *(3 points)*

Solution:

Using Gaussian Elimination, or simple inspection, we find that

$$\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

will do.

b. Find $\ker A$, and hence find the point $y \in \mathbb{R}^3$ with the smallest norm such that $Ay = b$. *(2 points)*

Solution:

The kernel is spanned by the vector

$$\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

Hence the solution set for $Ay = b$ is given by

$$\begin{bmatrix} -t \\ 2t \\ 3+t \end{bmatrix}$$

We want to find the closest point on this line to the origin. This will occur at the point on the line where the vector to the origin is orthogonal to the line itself. Thus we look for $t \in \mathbb{R}$ such that

$$\begin{bmatrix} -t \\ 2t \\ 3+t \end{bmatrix}^\top \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = 0$$

Expanding out, this gives $t + 4t + 3 + t = 0$, whence $t = -\frac{1}{2}$. The closest point is thus at

$$\begin{bmatrix} \frac{1}{2} \\ -1 \\ 2\frac{1}{2} \end{bmatrix}$$

c. Find a $z \in \mathbb{R}^3$ such that Az is the nearest point to c. *(3 points)*

Solution:

We solve this using the *method of least squares*. The problem is thus to solve $A^\top A z = A^\top c$. Expanding out, this is

$$\begin{bmatrix} 44 & 20 & 4 \\ 20 & 10 & 0 \\ 4 & 0 & 4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 60 \\ 30 \\ 0 \end{bmatrix}$$

Using Gaussian Elimination, we obtain a solution

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Problem 5.

Let $C([0,1], \mathbb{C})$ denote the space of continuous functions from $[0,1]$ to $\mathbb{C}$. Let

$$(f, g) = \int_0^1 f(t)\overline{g(t)}dt$$

be the standard inner product on $C([0,1], \mathbb{C})$.

a. Let $R\colon C([0,1],\mathbb{C}) \to C([0,1],\mathbb{C})$ be the linear transformation which sends $f \in C([0,1],\mathbb{C})$ to the function Rf given by $(Rf)(t) = f(1-t)$. Show that R is self-adjoint; that is, that

$$(Rf, g) = (f, Rg)$$

for all $f, g \in C([0,1],\mathbb{C})$. *(2 points)*

Solution:

We compute

$$\begin{aligned}
(Rf, g) &= \int_0^1 (Rf)(t)\overline{g(t)}dt \\
&= \int_0^1 f(1-t)\overline{g(t)}dt \\
&= \int_1^0 f(s)\overline{g(1-s)}(-ds) && s = 1-t \\
&= \int_0^1 f(s)\overline{g(1-s)}ds \\
&= \int_0^1 f(s)\overline{(Rg)(s)}ds \\
&= (f, Rg)
\end{aligned}$$

and so R is self-adjoint.

b. For $f \in C([0,1],\mathbb{C})$ prove that there are f_+ and f_- in $C([0,1],\mathbb{C})$ with $f = f_+ + f_-$, $Rf_\pm = \pm f_\pm$, and $(f_+, f_-) = 0$. *(3 points)*

Solution:

The key to this is to see that $R^2 = I$. Thus R acts a bit like conjugation, or taking adjoints, and for both of these we know how to do the analogous constructions. Thus we define $f_+ = \frac{1}{2}(f + Rf)$ and $f_- = \frac{1}{2}(f - Rf)$. Then $f_+ + f_- = f$, $Rf_+ = f_+$, $Rf_- = -f_-$, and

$$(f_+, f_-) = (Rf_+, -Rf_-) = -\left(R^2 f_+, f_-\right) = -(f_+, f_-)$$

whence $(f_+, f_-) = 0$.

c. Find a, b, c such that

$$\int_0^1 \left|4t^2 - 4t - a\sin(2\pi t) - b\cos(\pi t) - c\right|^2 dt$$

is minimal. *(3 points)*

Solution:

Let $f(t) = 4t^2 - 4t$. We know that to solve this, we need to project $f(t)$ onto the space spanned by $\sin(2\pi t)$, $\cos(\pi t)$, and 1. However, before computing the inner products, we note that we can use R to help us considerably.

Firstly, $(Rf)(t) = 4(1-t)^2 - 4(1-t) = 4t^2 - 4t = f(t)$. Secondly, $R\sin(2\pi t) = \sin(2\pi(1-t)) = -\sin(2\pi t)$. Thirdly, $R\cos(\pi t) = \cos(\pi(1-t)) = -\cos(\pi t)$. Hence $(f, \sin(2\pi t)) = 0$ and $(f, \cos(\pi t)) = 0$. Thus we only need to work out the inner product $(f, 1)$ which is

$$(f, 1) = \int_0^1 4t^2 - 4t dt = \left[\frac{4}{3}t^3 - 2t^2\right]_0^1 = \frac{26}{3}$$

Hence we have $a = 0$, $b = 0$, and $c = \frac{26}{3}$.