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English version

TMA4145 Linear Methods: Continuation Exam

16th August 2010
Time: 15:00–19:00
Examination Aids: D
No written and handwritten examination support materials are permitted.
Calculator: Citizen SR-270X or Hewlett Packard HP30S

Problem 1.

Answer any *four* of the following.

i. Give a definition of a continuous function from one metric space to another.
ii. Define what it means for a metric space to be *complete*.
iii. State the *rank theorem* (also called the *rank–nullity theorem*).
iv. In the $PA = LU$ factorisation of a matrix, what properties do the four matrices have?
v. Give a definition of an *orthonormal family* in an inner product space.
vi. Write out the *parallelogram identity* (also called the *parallelogram law*) for the norm in an inner product space.
vii. Define the L^2–norm on $C([0,1],\mathbb{C})$, the space of complex–valued continuous functions on $[0,1]$.

(8 points)

Problem 2.

Define $W \subseteq \mathbb{R}^5$ as the subspace:

$$W := \left\{ \begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} \text{ with } \begin{array}{rcrcrcrcrl} v & + & 3w & - & x & + & 3y & + & 5z & = 0, \\ 3v & + & 9w & - & 3x & + & 3y & + & 9z & = 0, \\ 2v & + & 6w & - & 2x & & & + & 4z & = 0, \\ v & + & 3w & - & x & - & 3y & - & z & = 0 \end{array} \right\}$$

Find the closest point in W to the vector:

$$\begin{bmatrix} 4 \\ 4 \\ -4 \\ -1 \\ 5 \end{bmatrix}$$

(8 points)

Solution:

There are several ways to do this; either by finding a basis for W, making it orthonormal, and computing the projection using it, or by using the method of least-squares.

For both, we start by finding a basis for W. To do this, we observe that W is the null space of the matrix

$$\begin{bmatrix} 1 & 3 & -1 & 3 & 5 \\ 3 & 9 & -3 & 3 & 9 \\ 2 & 6 & -2 & 0 & 4 \\ 1 & 3 & -1 & -3 & -1 \end{bmatrix}$$

and thus a basis is given by:

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

Hence W is the image of the matrix:

$$B := \begin{bmatrix} -3 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The method of least-squares says that we should solve $B^T B x = B^T b$. This expands to:

$$\begin{bmatrix} 10 & -3 & 6 \\ -3 & 2 & -2 \\ 6 & -2 & 6 \end{bmatrix} x = \begin{bmatrix} 8 \\ 0 \\ 2 \end{bmatrix}$$

Applying Gauss elimination yields the solution $\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$. Applying B to this produces the desired result:

$$\begin{bmatrix} 2 \\ -2 \\ -2 \\ -1 \\ 1 \end{bmatrix}$$

Problem 3.

Let $\|\cdot\|_2$ be the Euclidean norm on $\mathbb{R}^2$. Let $d\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be the function defined by:

$$d\big(\vec{x},\vec{y}\big) := \begin{cases} \left\|\vec{x}-\vec{y}\right\|_2 & \text{if } \mu\vec{x} = \lambda\vec{y} \text{ for some } \mu,\lambda \in \mathbb{R}, \\ \left\|\vec{x}\right\|_2 + \left\|\vec{y}\right\|_2 & \text{otherwise.} \end{cases}$$

1. Prove that d defines a metric on $\mathbb{R}^2$. *(4 points)*
   Solution:
   We need to check the axioms for a metric:
   (a) $d(\vec{x},\vec{y})$ is either a length or a sum of two lengths so is always a positive real number.
   (b) If $\vec{y} = \vec{x}$ then $d(\vec{x},\vec{y}) = \left\|\vec{x}-\vec{y}\right\|_2 = 0$; conversely, if $d(\vec{x},\vec{y}) = 0$ then either $\vec{x} = \vec{y}$ (if the first condition holds) or $\vec{x} = \vec{0} = \vec{y}$ (if the second).
   (c) Since $\left\|-\vec{v}\right\|_2 = \left\|\vec{v}\right\|_2$, $d(\vec{x},\vec{y}) = d(\vec{y},\vec{x})$.
   (d) There are several cases to consider for transitivity. If $\vec{x},\vec{y},\vec{z}$ are all collinear, then it holds since it holds for the Euclidean metric. If none are collinear, then it holds equally obviously. If $\vec{x}$ and $\vec{y}$ are collinear, we compare

   $$d(\vec{x},\vec{z}) = \left\|\vec{x}\right\|_2 + \left\|\vec{z}\right\|_2 \quad \text{with} \quad d(\vec{x},\vec{y}) + d(\vec{y},\vec{z}) = \left\|\vec{x}-\vec{y}\right\|_2 + \left\|\vec{y}_2\right\| + \left\|\vec{z}\right\|_2$$

   which gives transitivity by the triangle inequality for $\|\cdot\|_2$. Finally, if $\vec{x}$ and $\vec{z}$ are collinear, we compare

   $$d(\vec{x},\vec{z}) = \left\|\vec{x}-\vec{z}\right\|_2 \quad \text{with} \quad d(\vec{x},\vec{y}) + d(\vec{y},\vec{z}) = \left\|\vec{x}\right\|_2 + \left\|\vec{y}\right\|_2 + \left\|\vec{y}_2\right\| + \left\|\vec{z}\right\|_2$$

   which also gives transitivity by the triangle inequality for $\|\cdot\|_2$.

2. Does d come from a norm? Justify your answer. *(2 points)*
   Solution:
   d does not come from a norm as it is not translation-invariant:

   $$d(\vec{e}_1,\vec{e}_2) = 2, \qquad d(\vec{e}_1+\vec{e}_1,\vec{e}_2+\vec{e}_1) = 2 + \sqrt{2}$$

3. Let $I\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the identity map. Is either of $I\colon (\mathbb{R}^2,\|\cdot\|_2) \to (\mathbb{R}^2,d)$ or $I\colon (\mathbb{R}^2,d) \to (\mathbb{R}^2,\|\cdot\|_2)$ continuous? Justify your answer. *(2 points)*
   Solution:
   The first is continuous, the second not.
   For the first, we have:

   $$\left\|\vec{x}-\vec{y}\right\|_2 \le d(\vec{x},\vec{y})$$

   For the second, the sequence $\vec{e}_1 + {}^1\!/\!_n\vec{e}_2$ converges to $\vec{e}_1$ in $(\mathbb{R}^2,\|\cdot\|_2)$ but not in $(\mathbb{R}^2,d)$.

Problem 4.

Let Poly_1 be the space of polynomials of degree at most 1 with real coefficients. Define an inner product on Poly_1 by the formula:

$$\langle p,q\rangle := p(0)q(0) + p(1)q(1).$$

You may assume that this does define an inner product.

1. Define a linear function $\mathrm{Poly}_1 \to \mathbb{R}$ by $p(t) \mapsto \int_0^1 p(t)dt$. Find a polynomial $r(t) \in \mathrm{Poly}_1$ such that:

   $$\langle p(t), r(t)\rangle = \int_0^1 p(t)dt$$

   for all polynomials $p(t) \in \mathrm{Poly}_1$. *(2 points)*
   Solution:

Let $p_0(t) = 1 - t$ and $p_1(t) = t$. These are form an orthonormal basis for Poly_1. Then

$$r(0) = \langle p_0(t), r(t)\rangle = \int_0^1 p_0(t)dt = 1/2$$
$$r(1) = \langle p_1(t), r(t)\rangle = \int_0^1 p_1(t)dt = 1/2$$

Hence

$$\begin{aligned} r(t) &= 1/2 p_0(t) + 1/2 p_1(t) \\ &= 1/2 - 1/2 t + 1/2 t \\ &= 1/2 \end{aligned}$$

2. Let $D\colon \mathrm{Poly}_1 \to \mathrm{Poly}_1$ be the differentiation operator: $(Dp)(t) = p'(t)$. Find its *adjoint*. That is, find the operator $D^*\colon \mathrm{Poly}_1 \to \mathrm{Poly}_1$ which satisfies:

$$\langle Dp(t), q(t)\rangle = \langle p(t), D^*q(t)\rangle$$

for all $p(t), q(t) \in \mathrm{Poly}_1$. *(4 points)*
Solution:
For $p(t) = a + bt$, $Dp(t) = b$ so

$$\langle Dp(t), q(t)\rangle = bq(0) + bq(1) = b(q(0) + q(1)).$$

On the other hand, if $D^*q(t) = c + dt$, then

$$\langle p(t), D^*q(t)\rangle = ac + (a + b)(c + d) = a(2c + d) + b(c + d).$$

Equating these yields $2c + d = 0$ and $c + d = q(0) + q(1)$. Solving this leads to $c = -q(0) - q(1)$ and $d = 2q(0) + 2q(1)$. Thus

$$D^*q(t) = (q(0) + q(1))(2t - 1).$$

3. Find a polynomial $s(t) \in \mathrm{Poly}_1$ such that for all $p(t) \in \mathrm{Poly}_1$ $\langle p(t), s(t)\rangle = \int_0^1 Dp(t)dt$. *(2 points)*
Solution:
We have:

$$\int_0^1 Dp(t)dt = \langle Dp(t), r(t)\rangle = \langle p(t), D^*r(t)\rangle$$

and so $s(t) = D^*r(t)$. Thus $s(t) = (r(0) + r(1))(2t - 1) = 2t - 1$.

Problem 5.

Let A be the matrix

$$\begin{bmatrix} 0.5 & 0.25 \\ 0.25 & 0.875 \end{bmatrix}$$

1. Explain why A has a unique eigenvector in the positive quadrant with length 1.

That is, there is a unique vector $\begin{bmatrix} x \\ y \end{bmatrix}$ with $x, y \geq 0$ and $x^2 + y^2 = 1$ which is an eigenvector of A.

Note: you are not required to actually find this eigenvector. *(3 points)*
Solution:
As A is symmetric, it has an orthonormal basis of eigenvectors. Since A is not diagonal, the two eigenvalues are distinct and thus this orthonormal basis is almost unique: the only variations allowed are sign and order. Thus A has four eigenvectors of unit length and they are placed in $\mathbb{R}^2$ at right-angles to each other. Simply by inspection, at least one must be in the positive quadrant and the only way to get two is if they are in the x and y directions. However, this is not the case for this matrix and thus exactly one lies in the positive quadrant.

2. Show that if $\vec{x}$ is a vector in the positive quadrant (i.e. both components positive) then $A\vec{x}$ is also in the positive quadrant.

   What does this tell you about the eigenvalue of A corresponding to the eigenvector referred to in part 1? *(2 points)*

   Solution:

   A vector $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ is in the positive quadrant if and only if $x \geq 0$ and $y \geq 0$. Assume that this is the case. Let $\begin{bmatrix} u \\ v \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$. Then $u = 0.5x + 0.25y \geq 0$ and $v = 0.25x + 0.875y \geq 0$. Hence $A\vec{x}$ is in the positive quadrant.

   This means that the eigenvalue of A corresponding to the eigenvector from part 1 is the larger of the two eigenvalues of A.

3. Explain how one could use Banach's Fixed Point Theorem to find the eigenvector referred to in part 1.

   Note: this question is asking you to *set up* the problem. You do not have to check that all the conditions apply, but you should explain what the conditions are for this particular case, and you should note which conditions have already been verified by the previous steps.

   If, in your answer to part 1, you actually found the eigenvector, for this part pretend that you do not know it. *(3 points)*

   Solution:

   To find the eigenvector, we consider the map

$$\vec{x} \mapsto \frac{1}{\left\|A\vec{x}\right\|} A\vec{x}$$

   defined on that part of the unit circle that lies in the positive quadrant. This is a complete metric space (as a closed subspace of $\mathbb{R}^2$) and the above mapping is well-defined by the previous part of this question.

   Thus all that remains is to show that it is a contraction. That is, that there is some $0 \leq \alpha < 1$ such that for $\vec{x}, \vec{y}$ in this part of the unit circle,

$$\left\| \frac{1}{\left\|A\vec{x}\right\|} A\vec{x} - \frac{1}{\left\|A\vec{y}\right\|} A\vec{y} \right\| \leq \alpha \left\|\vec{x} - \vec{y}\right\|.$$