Part A

Question 1.

This question concerns sequences in metric spaces.

1. Give the definition of a sequence converging to a limit in a metric space. *(4 points)*
   Solution:
   Let (M, d) be a metric space. A sequence (s_n) in (M, d) converges to a limit, say $s \in M$, if for each $\epsilon > 0$ there is a $N \in \mathbb{N}$ such that whenever $n \geq N$, $d(s_n, s) < \epsilon$.
2. The sequence $(\frac{1}{n})$ can be considered as a sequence in each of the following metric spaces. For each, decide which of the following *best* describes the sequence:
   - Convergent
   - Cauchy
   - Neither convergent nor Cauchy

   (a) $\mathbb{R}$ with metric $d(s, t) = |s - t|$. *(2 points)*
   Solution:
   Convergent

   (b) $(0, 1)$ with metric $d(s, t) = |s - t|$. *(2 points)*
   Solution:
   Cauchy

   (c) $(0, 1)$ with metric $d(s, t) = |\frac{1}{s} - \frac{1}{t}|$ *(2 points)*
   Solution:
   Not convergent or Cauchy: we have $d(\frac{1}{n}, \frac{1}{m}) = |n - m|$.

Question 2.

This question concerns vector spaces.

1. Define what it means for a function from one vector space to another to be linear. *(4 points)*
   Solution:
   A function $T\colon V \to W$ is linear, where V and W are vector spaces, if $T(v + \lambda w) = Tv + \lambda Tw$ for all $v, w \in V$ and $\lambda \in \mathbb{R}$.
2. Which of the following are subspaces of $\mathbb{R}^3$?

   (a) $\{(x, y, z) \in \mathbb{R}^3 \mid 3x + 4y + z = 0\}$. *(2 points)*
   Solution:
   Subspace.

   (b) $\{(x, y, z) \in \mathbb{R}^3 \mid xy = z\}$. *(2 points)*
   Solution:
   Not a subspace.

   (c) $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 0\}$. *(2 points)*
   Solution:
   Subspace.

Question 3.

This question concerns functions between metric spaces.

1. Define what it means for a function from one metric space to another to be continuous. *(4 points)*
   Solution:
   Either: a function $f\colon (M_1, d_1) \to (M_2, d_2)$ is continuous if, whenever $U \subseteq M_2$ is open in (M_2, d_2) then $f^{-1}(U)$ is open in (M_1, d_1).
   Or: a function $f\colon (M_1, d_1) \to (M_2, d_2)$ is continuous if, for all $x \in M_1$ and $\epsilon > 0$ there is a $\delta > 0$ such that if $d_1(x, y) < \delta$ then $d_2(f(x), f(y)) < \epsilon$.
2. For each of the following functions between metric spaces decide which of the following *best* describes it:
   - Isometry.
   - Lipschitz.
   - Continuous.
   - None of the above.

(a) $f\colon (\mathbb{R}, d(s,t) = |s-t|) \to (\mathbb{R}, d(s,t) = |s-t|)$, $f(t) = t^2$. *(2 points)*
Solution:
Continuous.

(b) $f\colon ([0,1], d(s,t) = |s-t|) \to (\mathbb{R}, d(s,t) = |s-t|)$, $f(t) = e^t$. *(2 points)*
Solution:
Lipschitz.

(c) $f\colon ((0,1), d(s,t) = |\frac{1}{s} - \frac{1}{t}|) \to (\mathbb{R}, d(s,t) = |s-t|)$, $f(t) = t^{-1}$. *(2 points)*
Solution:
Isometry.

Part B

Question 1.

Let Poly_2 be the space of polynomials of degree at most 2. For $i \in \mathbb{N}$, let $p_i \in \mathrm{Poly}_2$ be the polynomial formed by expanding $(x+1)^i$ and throwing away all terms of degree higher than 2. Thus, $p_0(x) = 1$, $p_1(x) = x+1$, $p_3(x) = 3x^2 + 3x + 1$, $p_8(x) = 36x^2 + 8x + 1$.

1. Define a linear transformation $\mathbb{R}^3 \to \mathbb{R}^3$ as follows: map (a,b,c) to the coefficients (in the order $x^2, x, 1$) of the polynomial $ap_2(x) + bp_3(x) + cp_4(x)$. Write down the matrix of this transformation. *(4 points)*
Solution:
This transformation is

$$
\begin{aligned}
\begin{bmatrix} a \\ b \\ c \end{bmatrix} &\mapsto ap_2(x) + bp_3(x) + cp_4(x) \\
&= a(x^2 + 2x + 1) + b(3x^2 + 3x + 1) + c(6x^2 + 4x + 1) \\
&= (a + 3b + 6c)x^2 + (2a + 3b + 4c)x + (a + b + c) \\
&\mapsto \begin{bmatrix} a + 3b + 6c \\ 2a + 3b + 4c \\ a + b + c \end{bmatrix} \\
&= \begin{bmatrix} 1 & 3 & 6 \\ 2 & 3 & 4 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}
\end{aligned}
$$

2. A linear transformation $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ has matrix

$$
\begin{bmatrix} 1 & 5 & 10 & 10 \\ 2 & 5 & 4 & 1 \\ 2 & 7 & 9 & 5 \\ 2 & 9 & 16 & 14 \end{bmatrix}
$$

Find a vector $x \in \mathbb{R}^4$ such that

$$
Tx = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}
$$

(6 points)

Solution:

$$\begin{bmatrix} 1 & 5 & 10 & 10 & 1 \\ 2 & 5 & 4 & 1 & 0 \\ 2 & 7 & 9 & 5 & 0 \\ 2 & 9 & 16 & 14 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 5 & 10 & 10 & 1 \\ 0 & -5 & -16 & -19 & -2 \\ 0 & -3 & -11 & -15 & -2 \\ 0 & -1 & -4 & -6 & -1 \end{bmatrix}$$
$$\mapsto \begin{bmatrix} 1 & 5 & 10 & 10 & 1 \\ 0 & -1 & -4 & -6 & -1 \\ 0 & -3 & -11 & -15 & -2 \\ 0 & -5 & -16 & -19 & -2 \end{bmatrix}$$
$$\mapsto \begin{bmatrix} 1 & 5 & 10 & 10 & 1 \\ 0 & -1 & -4 & -6 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 4 & 11 & 3 \end{bmatrix}$$
$$\mapsto \begin{bmatrix} 1 & 5 & 10 & 10 & 1 \\ 0 & -1 & -4 & -6 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

So

$$\begin{aligned} x_4 &= 1, \\ x_3 &= 1 - 3x_4 = 1 - 3 = -2, \\ x_2 &= 1 - 6x_4 - 4x_3 = 1 - 6 + 8 = 3, \\ x_1 &= 1 - 5x_4 - 10x_3 - 10x_2 = 1 - 5 + 20 - 30 = -4. \end{aligned}$$

Question 2.

1. Let (M, d) be a metric space. Let $f\colon X \to M$ be an injective function (so if $f(x) = f(y)$ then $x = y$). Define $d_f\colon X \times X \to [0, \infty)$ by $d_f(x, y) = d(f(x), f(y))$. Prove that d_f is a metric. *(4 points)*

   Solution:

   (a) Let $x, y \in X$. If $d_f(x, y) = 0$ then $d(f(x), f(y)) = 0$. As (M, d) is a metric space, this means that $f(x) = f(y)$. Since f is injective, this means that $x = y$. Conversely, if $x = y$ then $f(x) = f(y)$ so $d_f(x, y) = d(f(x), f(y)) = 0$ since d is a metric on M.

   (b) Let $x, y \in X$. Then as d is symmetric, $d(f(x), f(y)) = d(f(y), f(x))$ and hence $d_f(x, y) = d_f(y, x)$.

   (c) Let $x, y, z \in X$. Then as d satisfies the triangle inequality and $f(x), f(y), f(z)$ are all points in M,
   $$d(f(x), f(z)) \leq d(f(x), f(y)) + d(f(y), f(z))$$
   and hence
   $$d_f(x, z) = d_f(x, y) + d_f(y, z).$$

2. *(You do not need to show working for this part.)* Let $f\colon \mathbb{R} \to \mathbb{R}^2$ be the function

   $$f(x) = \begin{cases} (x, 0) & \text{if } x \in \mathbb{Q}, \\ (x, 1) & \text{if } x \notin \mathbb{Q}. \end{cases}$$

   Define the metric d_f on $\mathbb{R}$ by
   $$d_f(x, y) = d_1(f(x), f(y)).$$
   Recall that d_1 is defined on $\mathbb{R}^2$ by $d_1((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$.
   For each of the following sequences, decide whether or not it converges in $(\mathbb{R}, d_f)$. If it converges, identify its limit. *(3 points)*

(a) $(1/n)$.
Solution:
This converges in $(\mathbb{R}, d_f)$ to 0.

(b) $(1/\sqrt{n})$.
Solution:
This does not converge in $(\mathbb{R}, d_f)$. We have $d(1/n^2, 1/(n^2+1)) \geq \frac{1}{2}$.

(c) $(1/\sqrt{n^2+1})$.
Solution:
This does not converge in $(\mathbb{R}, d_f)$ (though it is Cauchy).

The square roots are always assumed to be positive.

3. Consider the identity map on $\mathbb{R}$. Is this continuous as a map $(\mathbb{R}, d) \to (\mathbb{R}, d_f)$ or $(\mathbb{R}, d_f) \to (\mathbb{R}, d)$, where $d(s,t) = |s-t|$ is the usual metric? *(3 points)*
Solution:
It is not continuous as a map $(\mathbb{R}, d) \to (\mathbb{R}, d_f)$ since the sequence $(1/\sqrt{n})$ converges in $(\mathbb{R}, d)$ but does not in $(\mathbb{R}, d_f)$. It is continuous as a map $(\mathbb{R}, d_f) \to (\mathbb{R}, d)$ since if we define $p\colon \mathbb{R}^2 \to \mathbb{R}$ by $p(x,y) = x$ then $x = pf(x)$. Since $f\colon (\mathbb{R}, d_f) \to (\mathbb{R}^2, d_1)$ is continuous (even an isometry) and $p\colon (\mathbb{R}^2, d_1) \to (\mathbb{R}, d)$ is continuous (even Lipschitz), we deduce that the identity on $\mathbb{R}$ is continuous as a map $(\mathbb{R}, d_f) \to (\mathbb{R}, d)$.

Question 3.

1. Let $g\colon [0,\infty) \to [0,\infty)$ be the map $g(t) = \sqrt{t+1}$. Prove that g is a contraction. Here, $[0,\infty)$ is equipped with its standard metric. *(2 points)*
You may use the Mean Value Theorem without proof (that if f is differentiable on $[a,b]$ with continuous derivative then $f(b) - f(a) = f'(c)(b-a)$ for some $c \in [a,b]$).
Solution:
As g is continuously differentiable on $(-1,\infty)$, the mean value theorem states that for $0 \leq s < t$ in $\mathbb{R}$ there is some $r \in (s,t)$ such that
$$g(t) - g(s) = g'(r)(t-s).$$
From this we deduce that g is Lipschitz on $[s,t]$ with constant $\max\{|g'(r)| : r \in [s,t]\}$. Since $g'(x) = \frac{1}{2}(x+1)^{-\frac{1}{2}}$, and this is decreasing, we see that g is Lipschitz with constant $g'(0) = \frac{1}{2}$, hence is a contraction.

2. Explain why there is some $t_0 \in [0,\infty)$ such that $t_0^2 = t_0 + 1$. *(3 points)*
Solution:
The space $[0,\infty)$ with its standard metric is complete and hence Banach's Fixed Point Theorem applies. This states that as g is a contraction, it has a unique fixed point. This fixed point statisfies $t_0 = g(t_0) = \sqrt{t_0+1}$. Squaring, we find that $t_0^2 = t_0 + 1$.

3. Find $n \in \mathbb{N}_0$ such that if $n \leq t \leq n+1$ then $n \leq g(t) \leq n+1$. What does this tell you about the fixed point of g? *(2 points)*
Solution:
Firstly, as $g'(t)$ is positive, g is increasing. It is therefore sufficient to find n such that $g(n) \geq n$ and $g(n+1) \leq n+1$. Putting in some figures, we find $g(0) = 1$, $g(1) = \sqrt{2}$, $g(2) = \sqrt{3}$, and $g(3) = 2$. Hence $n = 1$ will do. This means that the fixed point of g lies in the interval $[1,2]$.

4. The equation $x^2 = x + 1$ has another solution which can be found as the fixed point of the map $t \mapsto -\sqrt{t+1}$. Show that this map defines a contraction on the interval $[-2/3, -5/9]$. (You may find it useful to know that $\sqrt{3} \simeq 1.7321$.) *(3 points)*
Solution:
Firstly, we observe that the map $t \mapsto -\sqrt{t+1}$ is well-defined on $[-2/3, -5/9]$ as a map into $\mathbb{R}$. For $t \in [-2/3, -5/9]$ we see that
$$\begin{gathered} -2/3 \leq t \leq -5/9 \\ 1/3 \leq t+1 \leq 4/9 \\ 1/\sqrt{3} \leq \sqrt{t+1} \leq 2/3 \\ -1/\sqrt{3} \geq -\sqrt{t+1} \geq -2/3 \end{gathered}$$

and thus $-\sqrt{t+1} \in [-2/3, -1/\sqrt{3}]$. Using the fact given in the question, we observe that $\sqrt{3} \leq 1.8$ and so $-1/\sqrt{3} \leq -5/9$.
Using the mean value theorem again, we see that the map $t \mapsto -\sqrt{t+1}$ is Lipschitz with Lipschitz constant

$$\max\{\frac{1}{2}(t+1)^{-\frac{1}{2}} : t \in [-2/3, -5/9]\}.$$

As this is decreasing, it takes its maximum at $t = -2/3$ where it is $\frac{1}{2}\sqrt{3} < 1$. Hence $t \mapsto -\sqrt{t+1}$ is a contraction on $[-2/3, -5/9]$.