Norges teknisk-naturvitenskapelig universitet
Institutt for matematiske fag

Suggested solutions
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Problem 1

a) Banach's Fixed Point Theorem: Let (X, d), $X \neq \emptyset$, be a complete metric space, and let $f : X \to X$ be a contraction. Then f has exactly one fixed point.

b) Since $X \neq \emptyset$ is complete and f^2 is a contraction, Banach's Fixed Point Theorem gives that f^2 has a unique fixed point $x^* \in X$. Since $f^2(f(x^*)) = f(f^2(x^*)) = f(x^*)$, the uniqueness gives that $f(x^*) = x^*$. Hence x^* is a fixed point of f as well. If $f(x) = x$, then also $f^2(x) = x$, and again by the uniqueness of x^*, $x = x^*$. Thus f has exactly one fixed point.

c) For $x, y \in C[0, 1]$ and $t \in [0, 1]$ we have

$$\begin{aligned}
|Fx(t) - Fy(t)| &= |\int_0^t (x(s) - y(s))ds| \\
&\leq \int_0^t |x(s) - y(s)|ds \\
&\leq (\int_0^t ds)d_\infty(x, y) \\
&= td_\infty(x, y).
\end{aligned}$$

This gives that

$$\begin{aligned}
|F^2x(t) - F^2y(t)| &\leq \int_0^t |Fx(s) - Fy(s)|ds \\
&\leq (\int_0^t sds)d_\infty(x, y) \\
&= \frac{1}{2}t^2 d_\infty(x, y) \\
&\leq \frac{1}{2}d_\infty(x, y).
\end{aligned}$$

Hence

$$d_\infty(F^2x, F^2y) = \max_{0\leq t\leq 1} |F^2x(t) - F^2y(t)| \leq \frac{1}{2}d_\infty(x, y),$$

and F^2 is a contraction on $C[0, 1]$ with the d_∞-metric. By **b)** F has a unique fixed point x^* since $(C[0, 1], d_\infty)$ is a complete metric space, and we can find x^* by iteration starting from any $x_0 \in C[0,1]$. Let $x_0 = 0$, and let $x_{n+1} = F^2 x_n$. Then

$$\begin{aligned} x_1(t) &= t + \frac{t^2}{2!} \\ x_2(t) &= t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} \\ &\vdots \end{aligned}$$

and we get by induction that

$$x_n(t) = t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^{2n}}{(2n)!}$$

for $n \geq 1$. Since d_∞-convergence implies convergence for each $t \in [0,1]$, we get that

$$x^*(t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} = e^t - 1.$$

(It is also true that $F^n x_0 \to x^*$ as $n \to \infty$.)

Problem 2

Here $A^T A = \left[\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right]$ and the eigenvalues are $\lambda_1 = 3$, $\lambda_2 = 1$ with corresponding orthonormal eigenvectors

$$v_{(1)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad , \quad v_{(2)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Hence

$$\Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Next:

$$u_{(1)} = \frac{1}{\sqrt{3}} A v_{(1)} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad u_{(2)} = \frac{1}{\sqrt{1}} A v_{(2)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Let

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

and solve $Bx = 0$. This gives $x = t(1, -1, 1)$, and we let

$$u_{(3)} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Then

$$U = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix},$$

and a singular value decomposition of A is $A = U \Sigma V^T$.

The pseudo inverse of A is then

$$A^+ = V \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} U^T = \frac{1}{3} \begin{bmatrix} 2 & 1 & -1 \\ -1 & 1 & 2 \end{bmatrix},$$

and the (unique) least squares solution of $Ax = (2, 1, 2)$ is

$$\hat{x} = A^+ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Problem 3

a) The characteristic polynomial of A is

$$P_A(\lambda) = \begin{vmatrix} 3-\lambda & -1 & -1 \\ 0 & 2-\lambda & 0 \\ 1 & -1 & 1-\lambda \end{vmatrix} = -(\lambda - 2)^3.$$

Thus $\lambda = 2$ is an eigenvalue of algebraic multiplicity 3. From

$$A - 2I = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we see that $\lambda = 2$ has geometric multiplicity 2. Hence a Jordan form of A is

$$J = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

The eigenvectors of A are

$$x = \begin{bmatrix} s+t \\ s \\ t \end{bmatrix}, \quad (s,t) \neq (0,0).$$

We must find $x_{(3)}$ such that $(A - 2I)x_{(3)} = x_{(2)}$ is an eigenvector. From

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & s+t \\ 0 & 0 & 0 & s \\ 1 & -1 & -1 & t \end{array}\right]$$

we see that this is possible if and only if $s = 0$, so let $s = 0$ and $t = 1$. Then we can use the solution

$$x_{(3)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

with

$$x_{(2)} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

We can then put $s = 1, t = 0$. This gives

$$x_{(1)} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

($x_{(1)}, x_{(2)}, x_{(3)}$ must be linearly independent.) Hence

$$S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

is such that $S^{-1}AS = J$.

b) The solution is

$$\begin{aligned} u &= e^{tA}u_0 \\ &= Se^{tJ}S^{-1}u_0 \\ &= S \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \qquad (c = S^{-1}u_0) \\ &= c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} 1+t \\ 0 \\ t \end{bmatrix} \end{aligned}$$

where $c_1, c_2, c_3 \in \mathbb{R}$.

Problem 4

a) Let $S_N = \sum_{n=1}^{N} \lambda_n e_n$ and $s_N = \sum_{n=1}^{N} |\lambda_n|^2$. For $M > N$ we then have (by Pythagoras' Theorem)

$$\|S_M - S_N\|^2 = \sum_{n=N+1}^{M} |\lambda_n|^2 = |s_M - s_N|.$$

Thus (S_N) is Cauchy if and only if (s_N) is Cauchy, and the claim follows since both H and $\mathbb{R}$ are complete.

b) Let $M = \text{span}\{1, t\} \subseteq L^2(0, 1)$ (with the usual abuse of notation). Then 1 and $\sqrt{3}(2t-1)$ is an orthonormal basis for M (here $\sqrt{3}(2t-1)$ is $t - \langle t, 1 \rangle$ normalized), and

$$\begin{aligned} \text{proj}_M e^t &= \langle e^t, 1 \rangle + \langle e^t, \sqrt{3}(2t-1) \rangle \sqrt{3}(2t-1) \\ &= e - 1 + 3\left(\int_0^1 e^t(2t-1)dt\right)(2t-1) \\ &= e - 1 + 3(3-e)(2t-1) \\ &= (4e - 10) + 6(3-e)t, \end{aligned}$$

hence $a = 4e - 10$ and $b = 6(3-e)$.

Problem 5

a) If $x \in C[0,1]$ and $y \in M$ we get

$$\begin{aligned}\|x-y\|^2 &= \int_0^1 |x(t)-y(t)|^2 dt \\ &= \int_0^{\frac{1}{2}} |x(t)|^2 dt + \int_{\frac{1}{2}}^1 |x(t)-y(t)|^2 dt \\ &\geq \int_0^{\frac{1}{2}} |x(t)|^2 dt.\end{aligned}$$

Let $x_n \in M$ such that $x_n \to x$ in $C[0,1]$. We show that $x \in M$. By the above

$$\int_0^{\frac{1}{2}} |x(t)|^2 dt \leq \|x - x_n\|^2 \to 0$$

as $n \to \infty$. Hence $\int_0^{\frac{1}{2}} |x(t)|^2 dt = 0$, and since x is continuous we must have $x(t) = 0$ for $0 \leq t \leq \frac{1}{2}$, i.e., $x \in M$. Thus M is closed.

b) If $x \in M$, then by **a)**

$$\|x-1\|^2 \geq \int_0^{\frac{1}{2}} dt = \frac{1}{2},$$

and $\|x-1\| \geq \frac{1}{\sqrt{2}}$. If $x_0 \in M$ with $\|x_0 - 1\|^2 = \frac{1}{2}$, then

$$\int_{\frac{1}{2}}^1 |x_0(t) - 1|^2 dt = \|x_0 - 1\|^2 - \int_0^{\frac{1}{2}} dt = 0$$

and $x_0(t) = 1$ for $\frac{1}{2} \leq t \leq 1$. This contradicts the continuity of x_0 and no such x_0 can exist.