



Problem 1

a) For $x, y \in C[0, 1]$ and $0 \leq t \leq 1$ we have

$$\begin{aligned} |(Gx)(t) - (Gy)(t)| &= \left| \int_0^t s(x(s) - y(s)) ds \right| \\ &\leq \int_0^t s|x(s) - y(s)| ds \\ &\leq \left(\int_0^t s ds \right) d_\infty(x, y) \\ &= \frac{1}{2} t^2 d_\infty(x, y) \leq \frac{1}{2} d_\infty(x, y). \end{aligned}$$

Hence $d_\infty(Gx, Gy) \leq \frac{1}{2} d_\infty(x, y)$ for all $x, y \in C[0, 1]$ showing that G is a contraction.

b) We prove this by induction. If $n = 1$, then $(Fx_0)(t) = \frac{t^2}{2} = \sum_{k=1}^1 (-1)^{k+1} \frac{t^{2k}}{2^k k!}$. Assume the formula holds for $(F^n x_0)(t)$, then

$$\begin{aligned} (F^{n+1} x_0)(t) &= \frac{t^2}{2} - \sum_{k=1}^n (-1)^{k+1} \int_0^t \frac{s^{2k+1}}{2^k k!} ds \\ &= \frac{t^2}{2} - \sum_{k=1}^n (-1)^{k+1} \frac{t^{2k+2}}{2^{k+1} (k+1)!} \\ &= \sum_{k=1}^{n+1} (-1)^{k+1} \frac{t^{2k}}{2^k k!}. \end{aligned}$$

Hence the formula holds for all $n \geq 1$.

c) Since $C[0, 1]$ with d_∞ is a complete metric space, and F is a contraction, it follows from Banach's Fixed Point Theorem that F has a unique point x^* . Moreover, x^* may be found by iteration: $x_0 = 0$ and $x_{n+1} = Fx_n$ ($n = 0, 1, 2, \dots$) gives $x^* = \lim x_n$. Pointwise x^* is by b) given by

$$x^*(t) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{t^{2k}}{2^k k!} = 1 - e^{-\frac{t^2}{2}}, \quad 0 \leq t \leq 1.$$

Problem 2

- a) $A^T A = 2 \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ has characteristic polynomial $-\lambda(\lambda - 2)(\lambda - 4)$ and hence eigenvalues $\lambda_1 = 4$, $\lambda_2 = 2$, $\lambda_3 = 0$. Thus the singular values are $\sigma_1 = 2$, $\sigma_2 = \sqrt{2}$ and

$$\Sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Corresponding orthonormal eigenvectors are

$$v_{(1)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_{(3)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

So let

$$V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

Then we let

$$u_{(1)} = \frac{1}{\sigma_1} A v_{(1)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad u_{(2)} = \frac{1}{\sigma_2} A v_{(2)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

In order to extend $u_{(1)}, u_{(2)}$ to an orthonormal basis for \mathbb{R}^4 , let

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

We get that $Bx = 0$ has the general solution

$$u = s \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}.$$

Since these two columns are orthogonal, we normalize and let

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}.$$

Then $A = U \Sigma V^T$ is a singular-value decomposition of A .

b) From the formula $A^+ = V\Sigma^+U^T$ we find the pseudo-inverse of A to be

$$\begin{aligned} A^+ &= \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 \\ 1 & 0 & 1 & 0 \end{bmatrix}. \end{aligned}$$

Problem 3

Let $M = \text{span}\{u, v\}$ where $u(t) = \frac{1}{\sqrt{\pi}} \cos t$ and $v(t) = \frac{1}{\sqrt{\pi}} \sin t$ in $L^2(0, 2\pi)$. Then u, v is an orthonormal basis for M , and a and b are given by the projection in $L^2(0, 2\pi)$ of x onto M . We have

$$\text{proj}_M x = \langle x, u \rangle u + \langle x, v \rangle v$$

where $\langle x, u \rangle = \int_0^{2\pi} x(t) \overline{u(t)} dt = \frac{1}{\sqrt{\pi}} \int_0^\alpha \cos t dt = \frac{1}{\sqrt{\pi}} \sin \alpha$, and $\langle x, v \rangle = \frac{1}{\sqrt{\pi}} (1 - \cos \alpha)$. Thus $a = \frac{\sin \alpha}{\pi}$ and $b = \frac{1 - \cos \alpha}{\pi}$.

Problem 4

Let $x = (x_n)_{n=1}^\infty \in \ell^2$ be given, i.e. $\sum_{n=1}^\infty |x_n|^2 < \infty$ where $x_n \in \mathbb{C}$, $n \geq 1$. Then let $x^{(n)} = (x_1, x_2, \dots, x_n, 0, 0, \dots)$ for $n \geq 1$. Then $x^{(n)} \in \ell_0$ and we have

$$\|x^{(n)} - x\|_2 = \left(\sum_{k=n+1}^\infty |x_k|^2 \right)^{1/2}$$

and since $\sum_{n=1}^\infty |x_n|^2$ converges, $\|x^{(n)} - x\|_2 \rightarrow 0$ as $n \rightarrow \infty$. Hence $x = \lim x^{(n)}$ and we see that $x \in \ell_0$. This shows that $\ell_0 = \ell^2$.

Problem 5

- a) $A \in \mathbb{C}^{n \times n}$ is normal if and only if $A^*A = AA^*$.
- b) If A is normal, then $\|A^*x\|^2 = \langle A^*x, A^*x \rangle = \langle AA^*x, x \rangle = \langle A^*Ax, x \rangle = \langle Ax, Ax \rangle = \|Ax\|^2$, and hence $\|A^*x\| = \|Ax\|$ for $x \in \mathbb{C}^n$.
- c) We have $(A - \lambda I)^*(A - \lambda I) = (A^* - \bar{\lambda}I)(A - \lambda I) = A^*A - \lambda A^* - \bar{\lambda}A + \lambda \bar{\lambda}I = AA^* - \lambda A^* - \bar{\lambda}A + \lambda \bar{\lambda}I = (A - \lambda I)(A^* - \bar{\lambda}I) = (A - \lambda I)(A - \lambda I)^*$. Hence $A - \lambda I$ is normal. Then we have

$$\begin{aligned} Ax = \lambda x &\Leftrightarrow (A - \lambda I)x = 0 \Leftrightarrow \|(A - \lambda I)x\| = 0 \\ &\Leftrightarrow \|(A - \lambda I)^*x\| = 0 \Leftrightarrow (A^* - \bar{\lambda}I)x = 0 \\ &\Leftrightarrow A^*x = \bar{\lambda}x. \end{aligned}$$

Problem 6

a) We need to show that for all $x, y, z \in \mathbb{R}$:

- (i) $d(x, y) \geq 0$, and $d(x, y) = 0 \Leftrightarrow x = y$
- (ii) $d(x, y) = d(y, x)$
- (iii) $d(x, y) \leq d(x, z) + d(z, y)$.

Since $f(0) = 0$ and f is strictly increasing the first part of (i) is clearly satisfied, and if $d(x, y) = f(|x - y|) = 0$, then $|x - y| = 0$ giving $x = y$. Since $|x - y| = |y - x|$, (ii) holds. From $f(x + y) \leq f(x) + f(y)$ for all x, y , we get (iii):

$$\begin{aligned}d(x, y) &= f(|x - y|) \\ &\leq f(|x - z| + |z - y|) && (f \text{ is increasing}) \\ &\leq f(|x - z|) + f(|z - y|) \\ &= d(x, z) + d(z, y).\end{aligned}$$

Thus d is a metric on \mathbb{R} .

b) Let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence in (\mathbb{R}, d) . Then to $\epsilon = 1$ there is an N such that if $m, n \geq N$, then $d(x_m, x_n) < 1$. Since $d(x, y) = |x - y| + 1 > 1$ if $x \neq y$, this means that $x_m = x_n$ for all $m, n \geq N$. Let $x = x_N$. Then clearly $x_n \rightarrow x$. Thus $(x_n)_{n=1}^{\infty}$ is convergent, and (\mathbb{R}, d) is complete.