TMA4145 Lineære metoder høsten 2007

Øving 12
Løsning

Problem 1

We note that $A^2 = 4I$, then $\lambda = \pm 2$ are the only possible eigenvalues. Let $\lambda_1 = 2$ and $\lambda_2 = -2$. Then

$$A - \lambda_1 I = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \text{ is an eigenvector.}$$

Since A is diagonalizable, $\lambda_2 = -2$ has geometric multiplicity 3, and also

$$E_{-2} = \text{span}\{x\}^{\perp} = \text{span}\{\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}\}.$$

Thus

$$Q^T A Q = \begin{bmatrix} 2 & & & \\ & -2 & & \\ & & -2 & \\ & & & -2 \end{bmatrix} \text{ if } Q = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}.$$

Problem 2

Here $P_A(\lambda) = \begin{vmatrix} 3-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (\lambda - 2)^2$, and we have $\lambda_1 = 2$ and $m_1 = 2$. We get $A - \lambda_1 I = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $x = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector. Let $y = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ is unitary, and

$$U^* A U = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} = T.$$

A is not diagonalizable since $d_1 = 1 < m_1 = 2$.

Problem 3

A basis $x_{(1)}, \ldots, x_{(d)}$ of $E_\lambda \subseteq F^n$ can always be extended to a basis $x_{(1)}, \ldots, x_{(n)}$ of F^n. With $S = [x_{(1)}|\ldots|x_{(n)}]$ we have

$$S^{-1} A S = \left[\begin{array}{c|c} \lambda I_d & B \\ \hline O & C \end{array}\right]$$

since $S^{-1}ASe_{(i)} = S^{-1}Ax_{(i)} = S^{-1}\lambda x_{(i)} = \lambda S^{-1}x_{(i)} = \lambda e_{(i)}$ if $1 \le i \le d$. Thus

$$P_A(t) = P_{S^{-1}AS}(t) = (\lambda - t)^d P_C(t)$$

and $d \le m$.

Problem 4

a) We have

$$J_{ij} = \begin{cases} \lambda & \text{if } j = i \\ 1 & \text{if } j = i+1 \\ 0 & \text{otherwise,} \end{cases}$$

and we prove that $(J^k)_{ij} = \binom{k}{j-i}\lambda^{k-j+i}$, $i \le j$, by induction on k. This is correct if $k = 1$, so assume the formula holds for $k \ge 1$. Then

$$\begin{aligned}
(J^{k+1})_{ij} &= \sum_{r=1}^{n} (J^k)_{ir} J_{rj} \qquad (J_{rj} = 0 \text{ if } r \ne j-1, j) \\
&= (J^k)_{i,j-1} + (J^k)_{ij}\lambda \\
&= \binom{k}{j-i-1}\lambda^{k-j+i+1} + \binom{k}{j-i}\lambda^{k-j+i+1} \\
&= [\binom{k}{j-i-1} + \binom{k}{j-i}]\lambda^{k-j+i+1} \\
&= \binom{k+1}{j-i}\lambda^{k+1-j+i},
\end{aligned}$$

and the formula follows by induction.

b) Using the formula in a) we get

$$\begin{aligned}
(e^{tJ})_{ij} &= \sum_k \frac{1}{k!}(J^k)_{ij} t^k \\
&= \sum_k \frac{1}{k!}\binom{k}{j-i}\lambda^{k-j+i} t^k \\
&= \sum_{0 \le j-i \le k} \frac{\lambda^{k-j+i}}{(j-i)!(k-j+i)!} t^k \qquad [l = k-j+i] \\
&= \left(\sum_l \frac{\lambda^l}{l!} t^l\right) \frac{t^{j-i}}{(j-i)!} \\
&= e^{\lambda t}\frac{t^{j-i}}{(j-i)!}, i \le j.
\end{aligned}$$

And $(e^{tJ})_{ij} = 0$ if $i > j$. Thus

$$e^{tJ} = e^{\lambda t} \begin{bmatrix} 1 & t & t^2/2! & \cdots & t^{n-1}/(n-1)! \\ & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & t^2/2! \\ & & & \ddots & t \\ 0 & & & & 1 \end{bmatrix}.$$

c)

$$u' = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} u \qquad \Rightarrow$$

$$u = e^{tJ} u_0 = e^{\lambda t} \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = c_1 e^{\lambda t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{\lambda t} \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix} + c_3 e^{\lambda t} \begin{bmatrix} \frac{t^2}{2} \\ t \\ 1 \end{bmatrix}.$$

Problem 5

a) We find the characteristic polynomial

$$P_A(\lambda) = \begin{vmatrix} 1-\lambda & 1 & \overset{\downarrow}{0} & 1 \\ 0 & 2-\lambda & 0 & 0 \\ -1 & 1 & 2-\lambda & 1 \\ -1 & 1 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 0 \\ -1 & 1 & 3-\lambda \end{vmatrix}$$
$$= (2-\lambda)^2 \big[(1-\lambda)(3-\lambda) + 1\big] = (2-\lambda)^4.$$

Hence the only eigenvalue is $\lambda = 2$ with $m = 4$. Since all the roots of $P_A(\lambda) = 0$ are real, A has a real Jordan form. From $(\lambda = 2)$

$$A - \lambda I = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 \\ -1 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we see that the geometric multiplicity is 3, and a Jordan form of A is

$$J = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

The eigenspace E_2 is given by

$$x = r \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad r, s, t \in \mathbb{R}.$$

We need to find $x_{(3)}$ and $x_{(4)}$ such that

1) $Ax_{(3)} = 2x_{(3)}$

2) $Ax_{(4)} = 2x_{(4)} + x_{(3)}$.

Then $x_{(3)} \in E_\lambda$, say $x_{(3)} = (r+t, r, s, t) \neq 0$. We solve for $x_{(4)}$ in 2):

$$\left[\begin{array}{cccc|c} -1 & 1 & 0 & 1 & r+t \\ 0 & 0 & 0 & 0 & r \\ -1 & 1 & 0 & 1 & s \\ -1 & 1 & 0 & 1 & t \end{array}\right] \sim \left[\begin{array}{cccc|c} -1 & 1 & 0 & 1 & r+t \\ 0 & 0 & 0 & 0 & r \\ 0 & 0 & 0 & 0 & -r+s-t \\ 0 & 0 & 0 & 0 & -r \end{array}\right].$$

This system has a solution if $r = 0$ and $s = t \neq 0$ ($x_{(3)} \neq 0$). Let $t = 1$ and $x_{(4)} = (-1, 0, 0, 0)$. Then $x_{(3)} = (1, 0, 1, 1)$. Then we need to choose two eigenvectors $x_{(1)}$ and $x_{(2)}$ such that $x_{(1)}$, $x_{(2)}$ and $x_{(3)}$ are independent. We can take

$$x_{(1)} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad x_{(2)} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Then

$$S = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

is such that $S^{-1}AS = J$ where

$$J = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

b) One finds

$$S^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 \end{bmatrix},$$

and

$$e^{tA} = Se^{tJ}S^{-1} = e^{2t} S \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix} S^{-1} = e^{2t} \begin{bmatrix} 1-t & t & 0 & t \\ 0 & 1 & 0 & 0 \\ -t & t & 1 & t \\ -t & t & 0 & 1+t \end{bmatrix}.$$

c) We have

$$u' = Au \quad \Rightarrow \quad u(t) = e^{tA}u_0 = e^{tA}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

$$= c_1 e^{2t}\begin{bmatrix} 1-t \\ 0 \\ -t \\ -t \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} t \\ 1 \\ t \\ t \end{bmatrix} + c_3 e^{2t}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_4 e^{2t}\begin{bmatrix} t \\ 0 \\ t \\ 1+t \end{bmatrix}.$$

Remark: If we had not been asked to find e^{tA}, we would simply write

$$u' = Au \quad \Rightarrow \quad u(t) = e^{tA}u_0 = Se^{tJ}S^{-1}u_0 = Se^{tJ}c \text{ etc.}$$

where $c = S^{-1}u_0$, and we need not find S^{-1}. This gives

$$u(t) = c_1 e^{2t}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{2t}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + c_4 e^{2t}\begin{bmatrix} t-1 \\ 0 \\ t \\ t \end{bmatrix}.$$

Problem 6

a) i) If $x \neq 0$, then $x^T(A+B)x = x^TAx + x^TBx > 0$, and $A + B > 0$.

ii) If A has eigenvalues $\lambda_1, \ldots, \lambda_n$, then A^k ($k \in \mathbb{Z}$) has eigenvalues $\lambda_1^k, \ldots, \lambda_n^k$, and $A > 0 \Rightarrow A^k > 0$.

iii) If A has eigenvalues $\lambda_1, \ldots, \lambda_n$, then e^A has eigenvalues $e^{\lambda_1}, \ldots, e^{\lambda_n}$ and they are all > 0. Since e^A clearly is symmetric, we have $e^A > 0$.

b) We have that $A = C^TC$ where

$$C = \begin{bmatrix} 1 & & & \mathbf{1} \\ & 1 & & \\ & & \ddots & \\ \mathbf{0} & & & 1 \end{bmatrix}$$

is non-singular, hence $A > 0$.

Problem 7

a) $A^TA = 2\left[\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{smallmatrix}\right]$ has characteristic polynomial $-\lambda(\lambda-2)(\lambda-4)$ and hence eigenvalues $\lambda_1 = 4$, $\lambda_2 = 2$, $\lambda_3 = 0$. Thus the singular values are $\sigma_1 = 2$, $\sigma_2 = \sqrt{2}$ and

$$\Sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Corresponding orthonormal eigenvectors are

$$v_{(1)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_{(3)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

So let

$$V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

Then we let

$$u_{(1)} = \tfrac{1}{\sigma_1} A v_{(1)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad u_{(2)} = \tfrac{1}{\sigma_2} A v_{(2)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

In order to extend $u_{(1)}, u_{(2)}$ to an orthonormal basis for $\mathbb{R}^4$, let

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

We get that $Bx = 0$ has the general solution

$$u = s \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}.$$

Since these two columns are orthogonal, we normalize and let

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}.$$

Then $A = U\Sigma V^T$ is a singular-value decomposition of A.

b) From the formula $A^+ = V\Sigma^+ U^T$ we find the pseudo-inverse of A to be

$$A^+ = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$