NTNU
Institutt for matematiske fag

TMA4145 LINEAR METHODS
Øving 7 – Obligatorisk øving

Wednesday, October 11, 2006
Time: 16:15 – 18:00

Permitted aids: Only HP30S.

Problem 1

a) (2 points.) Show that $d(x, y) = |e^{-x} - e^{-y}|$ defines a metric on $\mathbb{R}$.

b) (3 points.) Show that the sequence $(x_n)_{n=1}^{\infty}$ where $x_n = n$ is a Cauchy sequence in the metric space $(\mathbb{R}, d)$. Is $(\mathbb{R}, d)$ a complete metric space?

Problem 2

Let $F : C[0, 1] \to C[0, 1]$ be defined by

$$(Fx)(t) = 1 + \int_0^t sx(s)\mathrm{d}s,\ 0 \le t \le 1.$$

a) (4 points.) Show that F is a contraction if $C[0, 1]$ has the d_∞-metric. State (without proof) Banach's Fixed Point Theorem, and explain why F has a unique fixed point x^*.

b) (3 points.) Let $x_0 = 0$ and define x_n by $x_n = Fx_{n-1}$ for $n \ge 1$. Show that for $n \ge 1$

$$x_n(t) = \sum_{k=0}^{n-1} \frac{t^{2k}}{2^k k!},$$

and use this to find a formula for $x^*(t)$.

Problem 3

The 3×4-matrix A satisfies $PA = LU$ where

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}, \text{ and } U = \begin{bmatrix} 2 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

a) (2 points.) Find a basis for $\ker A = \mathbf{N}(A)$ and $\operatorname{im} A = \mathbf{C}(A)$.

b) (3 points.) Find a basis for $\operatorname{im} A^T = \mathbf{C}(A^T)$ and $\ker A^T = \mathbf{N}(A^T)$.

Problem 4

(3 points.) Find the Cholesky factorization $A = C^T C$ (C upper triangular with $c_{ii} > 0$) if

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 5 & 3 \\ 1 & 3 & 11 \end{bmatrix}.$$