NTNU
Department of Mathematical Sciences

Suggested solutions for TMA4145 LINEAR METHODS Midterm

English

Wednesday, October 10, 2007

Time: 17:15 – 19:00

Permitted aids: Code D (HP30S only).

Problem 1

a) Since $d(x,y) = |x-y|$ is a metric on $(0,\infty)$ we get for all $x,y,z \in X$

1) $\tilde{d}(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right| \geq 0$ for all $x,y \in X$; and $\tilde{d}(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right| = 0 \iff \frac{1}{x} = \frac{1}{y} \iff x = y$.

2) $\tilde{d}(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{1}{y} - \frac{1}{x}\right| = \tilde{d}(y,x)$.

3) $\tilde{d}(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right| \leq \left|\frac{1}{x} - \frac{1}{z}\right| + \left|\frac{1}{z} - \frac{1}{y}\right| = \tilde{d}(x,z) + \tilde{d}(z,y)$.

Thus $\tilde{d}$ is a metric on X, and $(X,\tilde{d})$ is a metric space.

b) If $x_n = 2^n$ then $\tilde{d}(x_m,x_n) = \left|\frac{1}{2^m} - \frac{1}{2^n}\right| \leq \frac{1}{2^m} + \frac{1}{2^n}$, and hence given $\epsilon > 0$ we have $\tilde{d}(x_m,x_n) < \epsilon$ if $m,n \geq N$ where $N > \log_2(\frac{2}{\epsilon})$. Thus $(x_n)_n$ is a Cauchy sequence in $(X,\tilde{d})$. If $x_n \to x$ in X, then $\tilde{d}(x_n,x) = \left|\frac{1}{2^n} - \frac{1}{x}\right| \to 0$ as $n \to \infty$. But $\left|\frac{1}{2^n} - \frac{1}{x}\right| \to \frac{1}{x} \neq 0$ as $n \to \infty$, and $(x_n)_n$ is not convergent. Hence $(X,\tilde{d})$ is not complete.

Problem 2

a) We have $|Fx(t) - Fy(t)| = |\int_0^t x(s)\mathrm{d}s - \int_0^t y(s)\mathrm{d}s| \leq \int_0^t |x(s)-y(s)|\mathrm{d}s \leq \int_0^t d_\infty(x,y)\mathrm{d}s = td_\infty(x,y)$, and hence $d_\infty(Fx,Fy) \leq ad_\infty(x,y)$. If $a<1$, then F is a contraction on $C[0,a]$.

Banach's Fixed Point Theorem: If $X \neq \emptyset$ is a complete metric space and $F\colon X \to X$ is a contraction, then F has a unique fixed point $x^* \in X$. (F is a contraction if there exists an $\alpha < 1$ such that $d(F(x),F(y)) \leq \alpha d(x,y)$ for all $x,y \in X$.)

Since $C[0,a]$ is complete and F is a contraction, F has a unique fixed point x^*.

b) From $x_1 = 1$ we get $x_2(t) = Fx_1(t) = 1 + t - \frac{t^2}{2}$ and $x_3(t) = Fx_2(t) = 1 - \frac{t^2}{2} + \int_0^t \left(1 + s - \frac{s^2}{2}\right)\mathrm{d}s = 1 - \frac{t^2}{2} + t + \frac{t^2}{2} - \frac{t^3}{3!} = 1 + t - \frac{t^3}{3!}$.

Assume by induction that $x_n = 1 + t - \frac{t^n}{n!}$ which is true for $n=1$ (and $n=2,3$). Then $x_{n+1}(t) = Fx_n(t) = 1 - \frac{t^2}{2} + \int_0^t \left(1 + s - \frac{s^n}{n!}\right)\mathrm{d}s = 1 - \frac{t^2}{2} + t + \frac{t^2}{2} - \frac{t^{n+1}}{(n+1)!} = 1 + t - \frac{t^{n+1}}{(n+1)!}$. Thus $x_n = 1 + t - \frac{t^n}{n!}$ for all $n \geq 1$. Let $x^*(t) = 1+t$. Then $d_\infty(x_n,x^*) = \max_t \left|\frac{t^n}{n!}\right| = \frac{a^n}{n!} \to 0$ as $n \to \infty$, and we see that $x^* = \lim_{n\to\infty} x_n$ in $C[0,a]$. We then know that x^* is the unique fixed point of F.

Problem 3

a) We have

$$U \sim \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and thus

$$\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

is a basis for $\ker A$. From

$$A = P^T LU = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -3 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -4 & -5 \\ 1 & -2 & -3 & -4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

we see that

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -3 \\ 1 \end{bmatrix}$$

is a basis for $\operatorname{im} A$.

b) A basis for $\operatorname{im} A^T$ is

$$\begin{bmatrix} 1 \\ -2 \\ -3 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

A basis for $\ker A^T$ is the last row of $L^{-1}P$ (since U has one row of zeroes). Since

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right],$$

$$L^{-1}P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix},$$

and a basis for $\ker A^T$ is

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

c) Since $y^T A = 0 \iff A^T y = 0$, any such y has the form

$$y = \alpha \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

where $\alpha \in \mathbb{R}$.
Since $[1\ 0\ 2]A = [1\ -2\ -2\ -3]$ we get the solution

$$z = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \ t \in \mathbb{R}.$$