Homework One

The questions in this homework are designed to help you get used to thinking about sets and functions in the right way. Remember that **sets** are determined by their **elements** and **functions** are determined by their **values**.

Question One

1. Consider the set $X = \{a, b, c, d\}$. Write out the *power set* of X, $\mathscr{P}(X)$; that is, list all the subsets of X. How many are there?

   How many subsets are there of each of $\varnothing$, $\{a\}$, $\{a, b\}$, and $\{a, b, c\}$? (You do not need to write out the power set in each case.)

   Make a conjecture about the size of $\mathscr{P}(Y)$ for a set, Y, of finite size, say n. Can you prove your conjecture?

2. Let us write $\mathbf{2} = \{0,1\}$. Let $X = \{a, b, c, d\}$. Write out the functions $X \to \mathbf{2}$. How many are there?

   How many functions are there from each of $\{a\}$, $\{a, b\}$, and $\{a, b, c\}$ to $\mathbf{2}$ are there?

   Make a conjecture relating the set of functions $Y \to \mathbf{2}$ and $\mathscr{P}(Y)$ for *any* set Y. Can you prove your conjecture?

3. Using your conjecture, translate the "algebra of subsets of X" (that is, unions, intersections, and complements) into the language of functions $Y \to \mathbf{2}$.

Question Two

If this question seems a little "abstract" to begin with, try it first for specific small sets, e.g. $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, and $C = \{c_1, c_2\}$.

1. Let A, B, and C be sets. Let $f : A \to B$ and $g : B \to C$ be functions. Recall that as the codomain of f is the domain of B, the composition $g \circ f : A \to C$ is well-defined. Let us write h for $g \circ f$.

   Let $F \subseteq A \times B$, $G \subseteq B \times C$, and $H \subseteq A \times C$ be the "graphs" of f, g, and h, respectively. That is, $F = \{(a, f(a))\}$, $G = \{(b, g(b))\}$, and $H = \{(a, h(a)\}$.

Write H in a form that depends explicitly on F and G but *not* explicitly on f, g, or h. That is, H as

$$H = \{(a, c) \in A \times C : \dots\}$$

where the dots are some condition on a and c which can involve F and G but not f, g, or h.

2. Let A, B, and C be sets. It is sometimes useful to think of a general subset $F \subseteq A \times B$ as some kind of *generalised function*. When we do so, we call it a **relation** or a **multifunction** from A to B.

   Let $F \subseteq A \times B$ and $G \subseteq B \times C$ be multifunctions. Give a way to construct a subset $H \subseteq A \times C$ so that if F and G correspond to *actual* functions f and g then H corresponds to the composition $g \circ f$.

   We refer to this subset H as the “composition” of G on F, $G \circ F$.

3. Let A, B be sets. Let $F \subseteq A \times B$ be a multifunction. Define $G \subseteq B \times A$ to be the “reverse” of F. That is:

   $$G = \{(b, a) : (a, b) \in F\}$$

   (Note: it is quite common to use the notation F^{-1} for what I am calling G.)

   Write down the “composition” $G \circ F \subseteq A \times A$.

   The “diagonal” in $A \times A$ is the subset

   $$\Delta(A) = \{(a, a) \in A \times A\}$$

   Similar we define the diagonal $\Delta(B) \subseteq B \times B$.

   What do each of the following statements imply about F? The first is done as an example.

   1. $\Delta(A) \subseteq G \circ F$

      This means that for each a there is some $b \in B$ with $(a, b) \in F$.

   2. $G \circ F \subseteq \Delta(A)$.

   3. $\Delta(B) \subseteq F \circ G$

3. $\Delta(B) \subseteq F \circ G$.

4. $F \circ G \subseteq \Delta(B)$.

Is there a combination of these statements that is equivalent to saying that F is a function?

Question Three

Let X be a set. How many functions $\emptyset \to X$ are there?

One of the ways of thinking about functions gives the answer to this straight away. However, to get full benefit from this question you should try to understand why the answer is what it is from *both* ways of thinking about functions. Also, you should think about the special case of $X = \emptyset$. To misquote Nils Bohr: if it doesn't make your head spin, you haven't understood it.