homework 11

Question One

Let A be an $m \times n$ matrix. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A^\top A$.

1. Express

$$\max\{\|A\vec{v}\|_2 : \|\vec{v}\|_2 = 1\}$$

   in terms of $\lambda_1, \ldots, \lambda_n$.

2. Let $\vec{v}$ be an eigenvector of $A^\top A$ with eigenvalue λ. Prove that $A\vec{v}$ is an eigenvector of $AA^\top$ and find it's eigenvalue. Let $\{\vec{u}_1, \ldots, \vec{u}_m\}$ be the basis for $\mathbb{R}^m$ that is used in the singular value decomposition of A. Explain why each $\vec{u}_j$ is an eigenvector of $AA^\top$.

3. Deduce (see homework 10) that the linear transformation $\vec{v} \mapsto A\vec{v}$ is continuous.

Question Two

1. Let A be the following matrix

$$A = \begin{bmatrix} 40 & -30 \\ -96 & 72 \\ 39 & 52 \end{bmatrix}$$

   Find the Singular Value Decomposition of A.

Question Three

Find the minimum polynomials of the following linear transformations.

1. $T\colon poly_3 \to poly_3$, $Tp(t) = p'(t)$.

2. $T\colon poly_3 \to poly_3$, $Tp(t) = p(t+1)$.

3. $T\colon \mathbb{R}^3 \to \mathbb{R}^3$, $T\vec{x} = A\vec{x}$ where A is the matrix

$$\begin{bmatrix} 4 & 2 & 8 \\ 5 & 0 & 3 \\ 4 & 0 & 0 \end{bmatrix}$$

Question Four

The usual norm on the space of $m \times n$ matrices is defined by

$$\|A\| = \max\{\|A\vec{v}\|_2 : \|\vec{v}\|_2 = 1\}$$

Note that this is well-defined by question one.

1. Prove that this is a norm.

2. Suppose that A is square and upper triangular. For $\epsilon > 0$, prove that there is a diagonal matrix B with $\|B\| < \epsilon$ such that $A+B$ has distinct eigenvalues.