Homework One

The questions in this homework are designed to help you get used to thinking about sets and functions in the right way. Remember that **sets** are determined by their **elements** and **functions** are determined by their **values**.

You should give reasons for your answers unless the question explicitly says otherwise.

Question One

Consider the following functions. Which of them are the same function?

1. $\mathbb{N} \to \mathbb{R}$, $n \mapsto \cos(n\pi)$
2. $\mathbb{N} \to \mathbb{R}$, $n \mapsto (-1)^n$
3. $\mathbb{N} \to \mathbb{R}$, $n \mapsto \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}$
4. $\mathbb{N} \to \mathbb{R}$, $n \mapsto \cos^n((2n+1)\pi)$
5. $\mathbb{N} \to \mathbb{R}$, $n \mapsto (\sin(n\pi) - 1)^n$

Question Two

1. Let $\{*\}$ be a set with one element in it. Let X be an arbitrary set. How many functions $X \to \{*\}$ are there?
2. Let X be an arbitrary set. How many functions $X \to \emptyset$ are there?
3. How many functions $\{x\} \to X$ are there?
4. *Something to ponder*[1] . There is a fair amount of similarity between $\{*\}$ and $\emptyset$. In lectures, we showed that there is only one empty set. How many sets with one element are there?

[1]That is, this is not part of the homework to be handed in.

Question Three

1. Draw $\{*, \bullet\} \times \{\times, \circ, \dagger\}$.

2. Draw the graphs (in $\{*, \bullet\} \times \{\times, \circ, \dagger\}$) of all the functions $\{*, \bullet\} \to \{\times, \circ, \dagger\}$.

3. Draw the graphs (in $\{\times, \circ, \dagger\} \times \{*, \bullet\}$) of all the functions $\{\times, \circ, \dagger\} \to \{*, \bullet\}$.

4. There is an obvious diagonal "flip" which takes $\{\times, \circ, \dagger\} \times \{*, \bullet\}$ to $\{*, \bullet\} \times \{\times, \circ, \dagger\}$. Under this flip are there any pairs of functions $f \colon \{\times, \circ, \dagger\} \times \{*, \bullet\}$ and $g \colon \{*, \bullet\} \times \{\times, \circ, \dagger\}$ such that the graph of f is "flipped" to the graph of g?

   (Given that you have just drawn the graphs, "by inspection" is sufficient reasoning for this part!)

5. Can you explain your answer to the previous part? That is, is there anything about the sets $\{*, \bullet\}$ and $\{\times, \circ, \dagger\}$ that tells you what the answer is without having to draw all the possible graphs?

6. Can you find two sets A and B and functions $f \colon A \to B$, $g \colon B \to A$ such that under the "flip" that takes $A \times B$ to $B \times A$, the graph of f is taken to the graph of g?

Question Four

Working with the "universal quantifier" ($\forall$) can sometimes be a little counter-intuitive. Ones intuition often breaks down at the two extremes: when working over the empty set there is often the feeling that it is "cheating" because there is nothing to do, and when working over some enormous set then there is often the worry that one hasn't quite checked all possibilities. What makes this even more confusing is that there are many synonyms for "for all" which mathematicians use[2] .

Working with the "existential quantifier" ($\exists$) is sometimes less scary as, in these cases, it suffices to find *one* thing satisfying the condition, and once

[2]this is because writing "for all" each time quickly makes a document hard to read. Mathematical documents are, first and foremost, intended to communicate something and so should conform to the standard rules and guidelines of style.

it's found then the case is solved and there's no issue about how many things there are.

Fortunately, statements with $\forall$ can be turned in to statements with $\exists$ with the help of a little negativity. Consider the following statement:

All dogs are cats.

We first "mathematise" this statement by introducing the "set of all dogs" and the "set of all cats" which we shall denote by Dog and Cat respectively. Then to say that "d is a dog" means "d is an element of Dog" which we write $d \in Dog$. So the above statement can be written:

$$\forall d \in Dog, d \in Cat$$

Now we transform this. The rule is that $\forall x \in X, \ldots$ turns in $\neg(\exists x \in X, \neg(\ldots))$. That is, there is a **double** negation. This is clearer in the example:

$$\neg(\exists d \in Dog : \neg(d \in Cat))$$

We usually write $\neg(d \in Cat)$ as $d \neq Cat$, so we could simplify this to:

$$\neg(\exists d \in Dog : d \neq Cat).$$

Translating this back to English, we arrive at:

It is not true that there is a dog which is not a cat.

It is now easy to see how to show that this statement is *false*: we simply exhibit a dog which is not a cat! If we cannot do that, our statement is true.

Now do the same with the following statements, including what you would have to do to show that the statement was false. Note the different ways that $\forall$ appears in each.

1. Each set has elements.
2. A positive number is not negative.
3. Every integer which has a rational square root is a perfect square.
4. Whenever $X \subseteq Y$ then there is a function $f : Y \to \{0, 1\}$ such that $X = \{x : f(x) = 1\}$