

Homework 12

The aim of this homework is to give you some familiarity with functionals on Hilbert spaces.

Question One

Let H be a Hilbert space (over \mathbb{C}). Let $f: H \rightarrow \mathbb{C}$ be a continuous linear functional. Let $w_f \in H$ be the corresponding element in H .

1. Let $SH := \{v \in H : \|v\| = 1\}$. Show that $f(SH)$ (the image of SH under f) is bounded in \mathbb{C} .
2. Show that there is a $v_0 \in SH$ such that $|f(v)| \leq f(v_0)$ for all $v \in SH$.
3. Show that the v_0 from the previous part is unique.

Question Two

The following properties are two of the most basic and most useful consequences of the existence of “enough” functionals.

Let H be a Hilbert space (over \mathbb{C}).

1. Let $u, v \in H$. Show that there is a continuous linear functional $f: H \rightarrow \mathbb{C}$ such that $f(u) \neq f(v)$.
2. Let $W \subseteq H$ be a closed subspace and let $v \in H$ be such that $v \notin W$. Show that there is a continuous linear functional $f: H \rightarrow \mathbb{C}$ such that $f(v) \neq 0$ whilst for all $w \in W$, $f(w) = 0$.

Question Three

The method of least squares works even if the codomain is infinite dimensional.

In this question, we shall solve the following problem:

Find $a, b, c \in \mathbb{C}$ which minimise the integral

$$\int_0^1 |a + bt + ct^2 - e^t|^2 dt \tag{1}$$

In this question, we work with $C([0, 1], \mathbb{C})$ with its standard inner product:

$$\langle f, g \rangle = \int_0^1 f(t) \overline{g(t)} dt$$

1. Let $\text{Poly}_2 \subseteq C([0, 1], \mathbb{C})$ be the subspace of polynomials of degree at most 2. Explain why finding the values a, b, c which minimise the integral in (1) is the same thing as finding the closest point in Poly_2 to e^t .
2. Suppose that $a, b, c \in \mathbb{C}$ are such that they minimise the integral in (1). Explain why

$$\int_0^1 t^k (\bar{a} + \bar{b}t + \bar{c}t^2) dt = \int_0^1 t^k e^t dt$$

for $k = 0, 1, 2$.

3. Set up a matrix equation for a, b, c .
4. Solve it and hence find the minimum value of the integral in (1).