Homework 9

The aim of this homework is to give you some familiarity with factorisation of linear transformations and matrices and with orthogonal families of vectors.

Question One

In this question you are to view differentiation as a linear map $\text{Poly}_3 \to \text{Poly}_3$.

1. Find a polynomial $p \in \text{Poly}_3$ such that

$$\begin{bmatrix} p'(0) \\ p'(1) \\ p'(2) \\ p'(3) \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 17 \\ 34 \end{bmatrix}$$

2. Find a polynomial $p \in \text{Poly}_3$ such that

$$\begin{bmatrix} p'(0) \\ p'(1) \\ p'(2) \\ p'(3) \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -11 \\ -26 \end{bmatrix}$$

3. Find a polynomial $p \in \text{Poly}_3$ such that

$$\begin{bmatrix} p'(0) \\ p'(1) \\ p'(2) \\ p'(3) \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 15 \\ 32 \end{bmatrix}$$

4. Find a polynomial $p \in \text{Poly}_3$ such that

$$\begin{bmatrix} p'(0) \\ p'(1) \\ p'(2) \\ p'(3) \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -19 \\ -47 \end{bmatrix}$$

Question Two

Apply the Gram-Schmidt algorithm to the set $\{1, t, t^2, t^3\}$ to produce an orthogonal family in Poly_3 with each of the following inner products.

(**Note:** you are asked to produce an **orthogonal** family, so you do not need to normalise at each step. You should make sure that you apply the algorithm correctly if you do not do the normalisation step.)

1. $\langle p, q\rangle = \int_{-1}^{1} p(t)q(t)dt$
2. $\langle p, q\rangle = \sum_{j=-1}^{2} p(j)q(j)$

Question Three

1. Let Q be a matrix whose columns form an orthonormal family. Prove that Q is injective. What might go wrong if we merely assumed that the columns of Q formed an *orthogonal* family?
2. Let Q be a matrix whose columns form an orthogonal family. Prove that Q^TQ is a diagonal matrix. What are the diagonal entries?
3. Let Q be a square matrix whose columns form an orthonormal family. Prove that Q is invertible. Identify its inverse.
4. Let A be a matrix with factorisation QR where the columns of Q form an orthonormal family and R is upper triangular. Explain why $A^TA = R^TR$.

Question Four

1. Let $\begin{bmatrix} a \\ b \end{bmatrix}$ be a unit vector in $\mathbb{R}^2$ with $a, b \in \mathbb{Q}$. Explain why there is some $k \in \mathbb{N}$ such that (ka, kb, k) are a Pythagorean triple.

   Find a unit vector orthogonal to $\begin{bmatrix} a \\ b \end{bmatrix}$ and hence find a matrix Q which has rational entries and whose columns form an orthonormal basis for $\mathbb{R}^2$.
2. Show that

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\}$$

is an orthogonal family. Find a fourth vector orthogonal to these three and write down the corresponding orthonormal family.