The Truncated Witten Genus

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Abstract

In this paper we define and examine the truncated Witten genus. It is defined as the equivariant index of the Dirac operator on the manifold $\operatorname{Map}(C_p, M)$ with its natural C_p-action. Here, $\operatorname{Map}(C_p, M)$ is the space of maps from the cyclic group of order p into a closed, connected, spin manifold, By applying the Atiyah-Singer index theorem we give a topological formula for the truncated Witten genus which is related to the formula for the Witten genus by truncation of the infinite products.

We also show that the equivariant index of the Dirac operator on the projective space $\mathbb{P}\operatorname{Map}(C_p, \mathbb{C}^{n+1})$ is closely related to the truncated Witten genus of $\mathbb{CP}^n$. The spaces $\mathbb{P}\operatorname{Map}(C_p, \mathbb{C}^{n+1})$ define a filtration of the space $\mathbb{P}\operatorname{Map}(S^1, \mathbb{C}^{n+1})$ which has been used to study equivariant objects on the smooth loop space of $\mathbb{CP}^n$.

1 Introduction

Let M be a closed, connected, spin manifold of even dimension d. Let p be a positive integer and let $C_p \subseteq S^1$ be the group of pth roots of unity; that is to say, the cyclic group of order p. We shall show in proposition 3.1 that the space of maps, $\operatorname{Map}(C_p, M)$, has a spin structure to which the natural C_p action on $\operatorname{Map}(C_p, M)$ lifts. Therefore the Dirac operator on $\operatorname{Map}(C_p, M)$ is C_p-equivariant. This operator is Fredholm which means that it has finite dimensional kernel and cokernel and thus these spaces are representations of C_p.

Definition 1.1. *The* Witten genus of M truncated at p, *written* $W_p(M)$, *is defined as follows: choose a generator* $\xi \in C_p$ *– that is, a primitive* p*th root of unity – and let*

$$W_p(M) := \operatorname{Index}_\xi \partial\!\!\!/_p = \operatorname{Tr} \xi|_{\ker \partial\!\!\!/_p} - \operatorname{Tr} \xi|_{\operatorname{coker} \partial\!\!\!/_p} .$$

We shall show in section 3 that this is independent of the choice of generator for C_p.

In [Wit88], Witten applied the equivariant Atiyah-Singer index theorem to the smooth[1] loop space, $\operatorname{Map}(S^1, M)$, of a manifold M and calculated that if

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[1] When considering maps from the circle we shall assume that these maps are smooth unless otherwise stated.

the Dirac operator on the loop space exists, its index at an element $\xi \in S^1$ of infinite order should be given by the formula:

$$\text{“}\operatorname{Index}_\xi \partial\!\!\!/_{\operatorname{Map}(S^1,M)}\text{”} = (-1)^{d/2}\xi^{-d/24}\left\langle \widehat{\mathcal{A}}(TM)\operatorname{ch}\left(\bigotimes_{k>0} S_{\xi^k}T_{\mathbb{C}}M\right), [M]\right\rangle .$$

This is a formal power series in ξ which we write as $W(M)(\xi)$. The notation used in this formula is explained in section 2. The factor $\xi^{-d/24}$ comes from the renormalisation of the infinite product $\prod_{k>0}\xi^k$ as $\xi^{-1/12}$. This is due to the interpretation of $\sum_{k>0} k$ as the value of the Riemann zeta function at -1, namely $-1/12$.

Using the identity in $K(M)[\![t]\!]$ that $S_tV = (\Lambda_{-t}V)^{-1}$, the expression in this formula can be rewritten as:

$$(-1)^{d/2}\xi^{-d/24}\left\langle \widehat{\mathcal{A}}(TM)\operatorname{ch}\left(\bigotimes_{k>0}(\Lambda_{-\xi^k}T_{\mathbb{C}}M)^{-1}\right), [M]\right\rangle .$$

In [Jon87], J.D.S. Jones showed that the family $\{\operatorname{Map}(C_p, M) : p \in \mathbb{N}\}$, with the natural C_p-action on $\operatorname{Map}(C_p, M)$, forms a *cocyclic object* with realisation the continuous loop space (which is homotopic to the smooth loop space, $\operatorname{Map}(S^1, M)$). For each p there is a natural map $\operatorname{Map}(S^1, M) \to \operatorname{Map}(C_p, M)$ defined by restricting a loop to the subgroup C_p. The family of these maps defines a functor from smooth loop spaces to cocyclic objects.

This observation has led to the principle that when studying objects on $\operatorname{Map}(S^1, M)$ which involve the circle action, one starts by studying the corresponding objects on $\operatorname{Map}(C_p, M)$ with the C_p-action. The definition of the truncated Witten genus clearly fits into this pattern. For more details on cocyclic models and specifically their relationship with loop spaces, we refer the interested reader to [Jon87].

In the case that p is odd, say $p = 2m + 1$, we shall apply the Atiyah-Singer index theorem to the Dirac operator on $\operatorname{Map}(C_{2m+1}, M)$ to prove the following theorem:

Theorem 1.2. *Let ξ be a primitive $2m + 1$th root of unity. The Witten genus truncated at $2m + 1$, $W_{2m+1}(M)$, is given by the formula:*

$$(-1)^{(m+1)d/2}\xi^{dm(m+1)/4}\left\langle \widehat{\mathcal{A}}(TM)\operatorname{ch}\left(\bigotimes_{k=1}^{m}(\Lambda_{-\xi^k}T_{\mathbb{C}}M)^{-1}\right), [M]\right\rangle .$$

Thus, up to sign and the renormalisation, the truncated Witten genus is obtained from the Witten genus by truncation of the infinite terms. As we shall note in section 3, the proof of this theorem is very similar to Witten's original computation in [Wit88].

In the theory of Hilbert manifolds it has been shown that a Hilbert manifold X contains a filtration, $\{X_n\}$, by finite dimensional closed manifolds such that $\lim_n X_n$ is homotopy equivalent to X; see for example [Muk70]. Thus one can calculate various aspects of X in terms of the family $\{X_n\}$.

To apply this to the study of the Witten genus via the truncated Witten genus we would require a filtration of $\operatorname{Map}(S^1, M)$ by the manifolds $\operatorname{Map}(C_p, M)$. Such a filtration does not exist. However, in the case of the loop space of projective space there is a related space which does have the right filtration.

Instead of $\operatorname{Map}(S^1, \mathbb{CP}^n)$ we consider $\mathbb{P}\operatorname{Map}(S^1, \mathbb{C}^{n+1})$. There is a restriction map from $\mathbb{P}\operatorname{Map}(S^1, \mathbb{C}^{n+1})$ to $\mathbb{P}\operatorname{Map}(C_p, \mathbb{C}^{n+1})$ which mirrors the map $\operatorname{Map}(S^1, \mathbb{CP}^n) \to \operatorname{Map}(C_p, \mathbb{CP}^n)$. Unlike the case of the loop space, these maps are split: there are maps $\mathbb{P}\operatorname{Map}(C_p, \mathbb{C}^{n+1}) \to \mathbb{P}\operatorname{Map}(S^1, \mathbb{C}^{n+1})$ such that the family of submanifolds is a filtration of $\mathbb{P}\operatorname{Map}(S^1, \mathbb{C}^{n+1})$. Moreover, the C_p-action on $\mathbb{P}\operatorname{Map}(C_p, \mathbb{C}^{n+1})$ is the restriction of an action by S^1 and the maps $\mathbb{P}\operatorname{Map}(C_p, \mathbb{C}^{n+1}) \to \mathbb{P}\operatorname{Map}(S^1, \mathbb{C}^{n+1})$ are S^1-equivariant.

The family of submanifolds obtained has the following description: it is the family of finite dimensional submanifolds, $\{\mathbb{P}(\mathbb{C}^{n+1}[z^{-1}, z]_a^b) : a < b\}$, where $\mathbb{C}^{n+1}[z^{-1}, z]_a^b$ is the space of Laurent polynomials in $\mathbb{C}^{n+1}$ of the form $v_a z^a + \cdots + v_b z^b$. The idea of using the space $\mathbb{P}\operatorname{Map}(S^1, \mathbb{C}^{n+1})$ and its filtration by the spaces $\mathbb{P}\mathbb{C}^{n+1}[z^{-1}, z]_a^b$ to study $\operatorname{Map}(S^1, \mathbb{CP}^n)$ has been used before: for example, in [CJS95] the authors demonstrate a connection between the Floer homologies of $\operatorname{Map}(S^1, \mathbb{CP}^n)$ and $\mathbb{P}\operatorname{Map}(S^1, \mathbb{C}^{n+1})$ and proceed to use the filtration to calculate this homology.

By applying the Atiyah-Singer index theorem to $\mathbb{P}\operatorname{Map}(C_{2m+1}, \mathbb{C}^{n+1})$ we shall prove the following theorem:

Theorem 1.3. *Let m, n be positive integers with n odd. The projective space $\mathbb{P}\operatorname{Map}(C_{2m+1}, \mathbb{C}^{n+1})$ is a spin manifold and the natural C_{2m+1}-action on it lifts to the spin structure. Let $\partial\!\!\!/$ be the corresponding Dirac operator. Let ξ be a primitive $(2m+1)$th root of unity; that is, a generator of C_{2m+1}. The equivariant index of $\partial\!\!\!/$ at ξ is given by the formula:*

$$\operatorname{Index}_\xi \partial\!\!\!/ = (-1)^m (2m+1) \xi^{m(m+1)/2} \left(\prod_{k=1}^{m} (1 - \xi^k)^{-2} \right) W_{2m+1}(\mathbb{CP}^n).$$

This paper is organised as follows: in section 2 we shall review the construction of the Dirac operator and the statement of the equivariant Atiyah-Singer index theorem. In section 3 we shall prove theorem 1.2. In section 4 we shall examine how the family $\{\mathbb{P}\operatorname{Map}(C_p, \mathbb{C}^{n+1})\}$ is related to the loop space of $\mathbb{CP}^n$. Finally, in section 5 we shall prove theorem 1.3 and explain why it is the expected result.

2 Dirac Operators and Index Theory

In this section we define the notation that we shall use in the proofs of theorems 1.2 and 1.3. We shall start by describing the various characteristic classes that we shall encounter. Then we shall review the basic construction of the Dirac operator on a spin manifold and state the equivariant Atiyah-Singer index theorem. Background on characteristic classes can be found in [MS74] and in [Hus94], on spin geometry in [LM89], and on the Atiyah-Singer index theorem in [LM89] and in [Sha78].

In this section, all manifolds, vector spaces, and vector bundles are assumed finite dimensional and smooth.

2.1 Characteristic Classes and Equivariant K-Theory

In our calculations we shall make use of three characteristic classes: the Chern character and the Todd class for complex vector bundles, and the Euler class for real oriented vector bundles. Of these we shall be most concerned with the Chern character.

For a commutative unital ring R we shall use the notation $H^\bullet(M;R)$ to denote the graded algebra of the cohomology of M with coefficients in R. That is, $H^\bullet(M;R) := \bigoplus_k H^k(M;R)$.

The Euler class of an oriented r-dimensional real vector bundle $E \to M$ is defined as follows: let E_0 denote the complement in E of the zero section. The orientation of E defines an element of $H^r(E,E_0;\mathbb{Z})$. The relative exact sequence for the pair (E,E_0) includes a map $H^r(E,E_0;\mathbb{Z}) \to H^r(E;\mathbb{Z})$. Since $E \to M$ is a homotopy equivalence, this defines a map $H^r(E,E_0;\mathbb{Z}) \to H^r(M;\mathbb{Z})$. The Euler class of E, $\mathrm{e}(E) \in H^r(M;\mathbb{Z})$, is the image of the element defined by the orientation of E under this map.

One useful tool when working with characteristic classes of complex bundles is the splitting principle; see for example [Hus94, ch 17]. This states that for a complex vector bundle $E \to M$ there is a manifold B and a proper map $f : B \to M$ with the following properties: firstly, $f^*E \to B$ decomposes as a sum of line bundles; secondly, $f^* : H^\bullet(M;\mathbb{Z}) \to H^\bullet(B;\mathbb{Z})$ is injective. This means that to describe a characteristic class for complex vector bundles it is sufficient to give its value on sums of line bundles.

The Chern character and Todd class can be described in terms of the first Chern class. For a line bundle $L \to M$, the first Chern class, $c_1(L) \in H^2(M;\mathbb{Z})$, is a characteristic class determined by the fact that its value on the canonical line bundle over $\mathbb{CP}^1$ is the volume form in $H^2(\mathbb{CP}^1;\mathbb{Z})$. The formulæ describing the Chern character, ch, and the Todd class, td, on sums of line bundles are:

$$\mathrm{ch}(L_1 \oplus \cdots \oplus L_r) = \sum_{j=1}^{r} e^{c_1(L_j)}, \quad \mathrm{td}(L_1 \oplus \cdots \oplus L_r) = \prod_{j=1}^{r} \frac{c_1(L_j)}{1 - e^{c_1(L_j)}}.$$

The functions appearing in these formulæ are to be interpreted as shorthand for the corresponding power series. Moreover, as M is a finite dimensional manifold, $c_1(L)$ is nilpotent for any line bundle and so, when evaluated, these power series are polynomials with rational coefficients.

If E and F are complex bundles which decompose as sums of line bundles then from the formula above, $\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F)$ and $\mathrm{ch}(E \otimes F) = \mathrm{ch}(E)\,\mathrm{ch}(F)$. The splitting principle implies that these identities hold for all complex bundles. Thus the Chern character is a homomorphism from the semiring of isomorphism classes of complex vector bundles to the cohomology ring of the base space. As such it extends to a ring homomorphism from the 0th degree K-theory of M, $K^0(M)$.

We shall also use the equivariant Chern character for manifolds with a trivial action of a finite cyclic group. Let $p \in \mathbb{N}$ and equip M with the trivial C_p-action. A C_p-equivariant complex vector bundle $V \to M$ decomposes according to the C_p-action as a sum $\bigoplus_{k=0}^{p-1} V_k$ where $\xi \in C_p$ acts on V_k by multiplication by ξ^k. For $\xi \in C_p$, the equivariant Chern character, ch_ξ, is defined as:

$$\operatorname{ch}_\xi(V) = \sum_{k=0}^{p} \xi^k \operatorname{ch}(V_k) \in H^\bullet(M; \mathbb{C}).$$

It is evident that the equivariant Chern character can also be defined as the composition of the map $K^0_{C_p}(M) \to K^0(M) \otimes \mathbb{C}$, $V \to \bigoplus \xi^k V_k$, with the ordinary Chern character extended over $\mathbb{C}$, $\operatorname{ch} : K^0(M) \otimes \mathbb{C} \to H^\bullet(M; \mathbb{C})$. It is also clear that the equivariant Chern character is a ring homomorphism.

For a complex vector bundle $V \to M$, the exterior algebra, $\Lambda V \in K^0(M)$, is defined as the sum of the exterior powers, $\Lambda^k V$. It is convenient to introduce the notation $\Lambda_\zeta V$ for $\zeta \in \mathbb{C}$. This denotes the element in $K^0(M) \otimes \mathbb{C}$ defined as:

$$\Lambda_\zeta V := \mathbb{C} \oplus \zeta V \oplus \zeta^2 \Lambda^2 V \oplus \cdots .$$

If $\xi \in C_p$ acts on V by multiplication by ζ then ξ acts on $\Lambda^k V$ by multiplication by ζ^k. It is thus evident that $\operatorname{ch}_\xi \Lambda V = \operatorname{ch} \Lambda_\zeta V$ and that $\operatorname{ch}_\xi \Lambda_{-1} V = \operatorname{ch} \Lambda_{-\zeta} V$.

2.2 The Dirac Operator

In this section, we give a brief review of the construction of the Dirac operator on a spin manifold.

Recall from [LM89, ch II, §2] that a *spin manifold* is an oriented Riemannian manifold with a spin structure on its tangent bundle and that the expression "M is spin" is taken to mean that $w_1(M) = 0$ and $w_2(M) = 0$; that is, the tangent bundle of M admits a spin structure for any choice of Riemannian metric and of orientation.

Let M be a closed, connected manifold. Let $E \to M$ be an oriented d-dimensional real vector bundle with an inner product which admits a spin structure; that is, the second Stiefel-Whitney class of E, $w_2(E) \in H^2(M; \mathbb{Z}_2)$, vanishes.

As E is oriented and has an inner product, the structure group of E reduces to SO_d. Let $P \to M$ be the principal SO_d-bundle associated to E. As E admits a spin structure, there is a principal Spin_d-bundle, $Q \to M$, together with a map of principal bundles $Q \to P$ which on fibres is the double cover $\mathrm{Spin}_d \to SO_d$. This Spin_d-bundle may not be unique.

Now suppose that d is even. The group Spin_d has two important complex irreducible unitary representations which we denote by Δ^+ and Δ^-. There is also a Spin_d-equivariant linear map $c : \mathbb{C}^d \otimes \Delta^+ \to \Delta^-$ called *Clifford multiplication.* Here Spin_d acts on $\mathbb{C}^d$ via the composition $\mathrm{Spin}_d \to SO_d \to U_d$. The *positive and negative spin bundles* of E are defined to be the bundles $\Delta^+(E) = \Delta^+_E := Q \times_{\mathrm{Spin}_d} \Delta^+$ and $\Delta^-(E) = \Delta^-_E := Q \times_{\mathrm{Spin}_d} \Delta^-$ respectively. The Clifford multiplication map is a well-defined fibrewise linear map, $c : E_\mathbb{C} \otimes \Delta^+_E \to \Delta^-_E$.

We shall not go into the details of the construction of the representations $\Delta^\pm$ of Spin_d. For details of this, see either [Sha78] for a brief introduction or [LM89] for a more in depth exposition. There are certain standard properties of the class in $K^0(M)$ represented by $\Delta^+_E - \Delta^-_E$ that we shall need which we list in the following proposition.

Proposition 2.1. *Let $E_1, E_2 \to M$ be even dimensional, oriented bundles which admit spin structures. Then $E_1 \oplus E_2$ is an even dimensional, oriented bundle which admits a spin structure. A choice of spin structure on each of E_1 and E_2 defines a choice of spin structure on $E_1 \oplus E_2$ such that:*

$$\Delta^+_{E_1 \oplus E_2} - \Delta^-_{E_1 \oplus E_2} = (\Delta^+_{E_1} - \Delta^-_{E_1}) \otimes (\Delta^+_{E_2} - \Delta^-_{E_2})$$

Let $E \to M$ be an even dimensional, oriented bundle which admits a spin structure. Let $F \to M$ be the same bundle with the opposite orientation. Then a spin structure on E defines a spin structure on F such that:

$$\Delta^+_F - \Delta^-_F = -(\Delta^+_E - \Delta^-_E).$$

Let $U \to M$ be a complex r-dimensional vector bundle and suppose that there is a line bundle $L \to M$ with $L \otimes L = \Lambda^r U$; equivalently, $2c_1(L) = c_1(U)$. Then $U_\mathbb{R}$ admits a spin structure and there is a choice of spin structure on $U_\mathbb{R}$ such that:

$$\Delta^+_{U_\mathbb{R}} - \Delta^-_{U_\mathbb{R}} = \overline{L} \otimes \Lambda_{-1} U = (-1)^r L \otimes \Lambda_{-1} \overline{U}. \tag{1}$$

In this last case we shall also need to consider the situation where there is a group which acts on the manifold and on the original vector bundle.

Proposition 2.2. *Let G be a finite group. Let M be a manifold with an action of G. Let $U \to M$ be a complex r-dimensional vector bundle with an action of G. Suppose that there is a complex line bundle $L \to M$ such that $L \otimes L = \Lambda^r U$. If L can be given an action of G to make this identity equivariant then the G-action on $U_\mathbb{R}$ lifts to the spin bundles of U.*

Proof. This is direct from equation (1) since $\Lambda_{-1} U$ inherits a natural action of the group. Therefore to specify an action on $\Delta^+_{U_\mathbb{R}} - \Delta^-_{U_\mathbb{R}}$ it is sufficient to specify the action on $\overline{L}$, or equivalently on L. ☐

Now let M be a spin manifold of even dimension d. Let $P \to M$ be the principal SO_d-bundle over M defined by the Riemannian metric. Let $Q \to M$ be the Spin_d-bundle corresponding to the choice of spin structure.

As $\mathrm{Spin}_d \to SO_d$ is a finite covering map, the Lie algebra of Spin_d is isomorphic to that of SO_d. Thus a connection on the principal SO_d-bundle, $P \to M$, lifts canonically to a connection on the principal Spin_d-bundle, $Q \to M$. Applying this lift to the Levi-Civita connection on P yields the spin connection on Q. This defines a covariant differentiation operator $\nabla : \Gamma(\Delta_M^+) \to \Gamma(T_{\mathbb{C}}^*M \otimes \Delta_M^-)$. We identify T^*M with TM via the metric and make the definition:

Definition 2.3. *The* Dirac operator *on M, $\partial\!\!\!/ : \Gamma(\Delta_M^+) \to \Gamma(\Delta_M^-)$, is the composition $c \circ \nabla$. That is:*

$$\partial\!\!\!/ : \Gamma(\Delta_M^+) \xrightarrow{\nabla} \Gamma(T_{\mathbb{C}}^*M \otimes \Delta_M^+) \xrightarrow{c} \Gamma(\Delta_M^-).$$

We now consider the equivariant situation. Let G be a finite group acting on M by orientation preserving diffeomorphisms. By altering the metric on M if necessary, we can assume that G acts by isometries on M. Thus the action of G on M lifts to an action on P. The uniqueness of the Levi-Civita connection means that it is preserved by the G-action. Let $Q \to M$ be a choice of spin structure on M. If the action of G on P lifts to an action on Q then G acts on the spin bundles $\Delta_M^\pm$. The Clifford multiplication map is G-equivariant as is the spin connection. Thus if the action of G on P lifts to Q, the Dirac operator on M is G-equivariant.

2.3 The Equivariant Atiyah-Singer Index Theorem

Let M be a closed, oriented, Riemannian manifold which is spin and is of even dimension d. Suppose that M has an action of the cyclic group C_p which lifts to a choice of spin structure on M. This means that the Dirac operator, $\partial\!\!\!/$, on M is C_p-equivariant and so its kernel and cokernel are representations of C_p. Their characters are functions $C_p \to \mathbb{C}$ which we combine to define the equivariant index of $\partial\!\!\!/$:

Definition 2.4. *The* equivariant index *of the Dirac operator on M is the function $C_p \to \mathbb{C}$ defined by:*

$$\xi \to \mathrm{Index}_\xi\, \partial\!\!\!/ := \mathrm{Tr}\, \xi|_{\ker \partial\!\!\!/} - \mathrm{Tr}\, \xi|_{\mathrm{coker}\, \partial\!\!\!/}\,.$$

The equivariant index theorem gives a formula for this function in terms of topological data of the manifold:

Theorem 2.5 (Atiyah-Singer). *For $\xi \in C_p$,*

$$\mathrm{Index}_\xi\, \partial\!\!\!/ = (-1)^{l/2} \left\langle \frac{\mathrm{ch}_\xi\, j^*(\Delta_M^+ - \Delta_M^-)\, \mathrm{td}(T_{\mathbb{C}} M^\xi)}{\mathrm{ch}_\xi(\Lambda_{-1} N_{\mathbb{C}}^\xi)\, \mathrm{e}(TM^\xi)}, [M^\xi] \right\rangle;$$

where: M^ξ is the fixed point set of the action of ξ on M, assumed to be orientable and of even dimension l; $j : M^\xi \to M$ is the inclusion; and $N^\xi \to M^\xi$ is the equivariant normal bundle to this inclusion. If M^ξ is not connected, this expression is summed over all components.

To simplify the expression from theorem 2.5, we observe that $j^*TM = TM^\xi \oplus N^\xi$ as equivariant bundles over M^ξ. We assume that this is a splitting of bundles admitting spin structures and thus from proposition 2.1:

$$j^*(\Delta_M^+ - \Delta_M^-) \cong (\Delta_{M^\xi}^+ - \Delta_{M^\xi}^-) \otimes (\Delta_{N^\xi}^+ - \Delta_{N^\xi}^-).$$

As the ξ action on TM^ξ is trivial, the ξ action on $\Delta_{M^\xi}^\pm$ is also trivial. Putting this into the expression in theorem 2.5, suppressing the summation over the components of M^ξ, yields:

$$\operatorname{Index}_\xi \partial\!\!\!/ = (-1)^{l/2} \left\langle \frac{\operatorname{ch}(\Delta_{M^\xi}^+ - \Delta_{M^\xi}^-)\operatorname{td}(T_\mathbb{C}M^\xi)}{\operatorname{e}(TM^\xi)} \frac{\operatorname{ch}_\xi(\Delta_{N^\xi}^+ - \Delta_{N^\xi}^-)}{\operatorname{ch}_\xi \Lambda_{-1} N_\mathbb{C}^\xi}, [M^\xi] \right\rangle.$$

The first part of this expression is the characteristic class $\widehat{\mathcal{A}}(TM^\xi)$ which occurs in the formula for the non-equivariant index of the Dirac operator over M^ξ. Thus:

$$\operatorname{Index}_\xi \partial\!\!\!/ = (-1)^{l/2} \left\langle \widehat{\mathcal{A}}(TM^\xi) \frac{\operatorname{ch}_\xi(\Delta_{N^\xi}^+ - \Delta_{N^\xi}^-)}{\operatorname{ch}_\xi \Lambda_{-1} N_\mathbb{C}^\xi}, [M^\xi] \right\rangle. \tag{2}$$

We can simplify this further when N is the underlying real bundle of a complex bundle.

Theorem 2.6. *Suppose that p is odd and that N^ξ is equivariantly the underlying real vector bundle of an r-dimensional complex vector bundle U with $c_1(U) = 0$. Let ζ be the pth root of unity such that ξ acts on $\Lambda^r U$ by fibrewise multiplication by ζ. Then:*

$$\operatorname{Index}_\xi \partial\!\!\!/ = (-1)^{r+(l/2)} \zeta^{\frac{1}{2}} \left\langle \widehat{\mathcal{A}}(TM^\xi) \left(\operatorname{ch}_\xi \Lambda_{-1} U\right)^{-1}, [M^\xi] \right\rangle,$$

where $\zeta^{\frac{1}{2}}$ is the unique pth root of unity which squares to ζ.

Proof. The bundle $\Lambda^r U$ is a line bundle with fibrewise group action. Therefore the action of ξ is multiplication by some pth root of unity, ζ (not necessarily primitive). As p is odd, there is a unique pth root of unity which squares to ζ (the other square root of ζ is not a pth root of unity). Write this as $\zeta^{\frac{1}{2}}$.

Because $c_1(U) = 0$, $\Lambda^r U$ is a trivial line bundle. Therefore its equivariant square root is a trivial line bundle with fibrewise action where ξ acts by multiplication by $\zeta^{\frac{1}{2}}$. Applying the equivariant Chern character to equation (1) yields:

$$\operatorname{ch}_\xi \left(\Delta_{N^\xi}^+ - \Delta_{N^\xi}^-\right) = (-1)^r \zeta^{\frac{1}{2}} \operatorname{ch}_\xi \Lambda_{-1} \overline{U}.$$

The process of taking the exterior power converts a sum of bundles into a product of their exterior powers. Thus as $U_{\mathbb{R}\mathbb{C}} = U \oplus \overline{U}$, $\operatorname{ch}_\xi \Lambda_{-1} N_\mathbb{C}^\xi = (\operatorname{ch}_\xi \Lambda_{-1} U)(\operatorname{ch}_\xi \Lambda_{-1} \overline{U})$. Substituting these expressions into equation 2 and cancelling the terms $\operatorname{ch}_\xi \Lambda_{-1} \overline{U}$ yields:

$$\operatorname{Index}_\xi \partial\!\!\!/ = (-1)^{r+(l/2)} \zeta^{\frac{1}{2}} \left\langle \widehat{\mathcal{A}}(M^\xi) \left(\operatorname{ch}_\xi \Lambda_{-1} U\right)^{-1}, [M^\xi] \right\rangle.$$

□

3 The Dirac Operator on Product Space

Let M be a closed, connected, oriented, Riemannian, spin manifold of even dimension d. Let G be a cyclic group, either C_p for some $p \in \mathbb{N}$ or S^1.

The proof of theorem 1.2 is similar to the method employed by Witten in [Wit88] in the original construction of the Witten genus. In both cases the space under consideration is the space of maps, $\operatorname{Map}(G, M)$, with its natural G-action. Before examining the formula from the Atiyah-Singer index theorem, we shall first show that in the case that G is finite, $\operatorname{Map}(G, M)$ is a spin manifold.

Let $\Sigma(C_p)$ be the symmetric group of C_p, considered as a set. We can think of $\Sigma(C_p)$ as $\operatorname{Diff}^+(C_p)$ for comparison with the case of the circle. The action of C_p on itself defines a monomorphism $C_p \to \Sigma(C_p)$. The manifold $\operatorname{Map}(C_p, M)$ has a natural $\Sigma(C_p)$-action given by precomposition. This action restricted to $C_p \subseteq \Sigma(C_p)$ agrees with the natural C_p-action.

Proposition 3.1. *There is a spin structure on* $\operatorname{Map}(C_p, M)$ *with a lift of the* $\Sigma(C_p)$*-action from* $\operatorname{Map}(C_p, M)$*.*

Proof. Let $P \to M$ be the oriented, orthonormal frame bundle of the tangent bundle of M, so it is an SO_d-principal bundle, and let $Q \to M$ be the spin structure of M.

The corresponding oriented, orthonormal frame bundle of the tangent bundle of $\operatorname{Map}(C_p, M)$ has the following description: it is $\operatorname{Map}(C_p, P) \times_\rho SO(\operatorname{Map}(C_p, \mathbb{R}^d))$ where $\rho : \operatorname{Map}(C_p, SO_d) \to SO(\operatorname{Map}(C_p, \mathbb{R}^d))$ is the natural group monomorphism. This monomorphism is covered by a group homomorphism $\sigma : \operatorname{Map}(C_p, \operatorname{Spin}_d) \to \operatorname{Spin}(\operatorname{Map}(C_p, \mathbb{R}^d))$. The bundle $\operatorname{Map}(C_p, Q) \times_\sigma \operatorname{Spin}(\operatorname{Map}(C_p, \mathbb{R}^d))$ is a spin bundle for $\operatorname{Map}(C_p, M)$.

The group $\Sigma(C_p)$ acts on $\operatorname{Map}(C_p, Q)$ by precomposition. Therefore it acts on the spin structure of $\operatorname{Map}(C_p, M)$. □

Corollary 3.2. *The truncated Witten genus is independent of the choice of generator of* C_p*.*

Proof. The Dirac operator on $\operatorname{Map}(C_p, M)$ is $\Sigma(C_p)$-equivariant whence its equivariant index is a class function $\Sigma(C_p) \to \mathbb{C}$. As such, it is constant on conjugacy classes in $\Sigma(C_p)$ and thus gives the same answer for any generator of C_p. □

We now turn to consideration of the equivariant Atiyah-Singer index theorem. We return to the general case where G is either C_p for $p \in \mathbb{N}$ or is S^1. In the case that G is S^1 the following argument is formal and is that of Witten in [Wit88]. To apply the index theorem, we need to determine the fixed point set and the normal bundle to its inclusion. The fixed point set of the G-action on $\operatorname{Map}(G, M)$ is simple to describe: it is a copy of M considered as the set of constant maps.

Proposition 3.3. *The normal bundle of the inclusion of the fixed point submanifold has the description:*

$$N = TM \otimes \operatorname{Map}(G, \mathbb{R})_0$$

where $\operatorname{Map}(G, \mathbb{R})_0$ *is the* G*-invariant complement of the constant maps in* $\operatorname{Map}(G, \mathbb{R})$*.*

Proof. Let $TM \to M$ be the tangent bundle of M. The total tangent space of $\operatorname{Map}(G, M)$ is $\operatorname{Map}(G, TM)$, the space of maps from G into the total space of the tangent space of M. The fibre of this at a point x in the fixed point set is $\operatorname{Map}(G, T_x M)$.

The inclusion $M \to \operatorname{Map}(G, M)$ as the constant maps defines an inclusion $TM \to T(\operatorname{Map}(G, M))$. In the description of $T(\operatorname{Map}(G, M))$ as $\operatorname{Map}(G, TM)$, this inclusion is also of the constant maps. Thus the fibrewise map $T_x M \to \operatorname{Map}(G, T_x M)$ at x in the fixed point set is as the inclusion of the subspace of constant maps. Thus the normal bundle at x is the G-invariant complement of $T_x M$ in $\operatorname{Map}(G, T_x M)$.

For any real finite dimensional vector space V there is a natural isomorphism $\operatorname{Map}(G, V) \cong V \otimes \operatorname{Map}(G, \mathbb{R})$ and the G-invariant complement of the constant maps is $V \otimes \operatorname{Map}(G, \mathbb{R})_0$. Thus the normal bundle to the inclusion of the fixed point set is as described in the statement. $\square$

We now restrict to the case where G does not have even order. Thus G is either of odd order, say $p = 2m+1$, or is S^1. This ensures that the non-trivial real representations of G are two dimensional and each is the underlying real space of a non-trivial complex irreducible representation of G. Hence $\operatorname{Map}(G, \mathbb{R})_0$ is the underlying real space of a complex representation of G. We choose such a representation, $\operatorname{Map}(G, \mathbb{C})_+$, and write:

$$N = TM \otimes (\operatorname{Map}(G, \mathbb{C})_+)_{\mathbb{R}} = (TM \otimes \operatorname{Map}(G, \mathbb{C})_+)_{\mathbb{R}}.$$

In the case that G is finite we need to be concerned with orientations. For an oriented real vector space V and a complex vector space U we wish to compare the orientations of $(V \otimes U)_{\mathbb{R}}$ with $V \otimes U_{\mathbb{R}}$. Let $(v_1, \dots, v_r)$ be an oriented basis for V, $(u_1, \dots, u_t)$ a (complex) basis for U. Then $(V \otimes U)_{\mathbb{R}}$ and $V \otimes U_{\mathbb{R}}$ have respective ordered bases:

$$\begin{gathered}(v_1 \otimes u_1, v_2 \otimes u_1, \dots, v_r \otimes u_1, v_1 \otimes iu_1, \dots, v_r \otimes iu_t),\\ (v_1 \otimes u_1, v_1 \otimes iu_1, v_2 \otimes u_1, \dots, v_r \otimes u_t, v_r \otimes iu_t).\end{gathered}$$

The permutation taking the first to the second has sign $(-1)^{tr(r-1)/2}$. If r is even, this simplifies to $(-1)^{tr/2}$.

We return to the normal bundle. The complex bundle of which this is the underlying real bundle is itself a complexification of a real bundle. Thus its first Chern class vanishes and theorem 2.6 applies. To apply it, we need to determine the group action on the top exterior power. In the infinite case, this is a purely formal argument.

Let $\mathbb{C}[k]$ denote the representation of G on $\mathbb{C}$ where $\lambda \cdot \zeta = \lambda \zeta^k$ for $\zeta \in G$. A choice for $\operatorname{Map}(G, \mathbb{C})_+$ is the following: if $G = C_p$, take $\operatorname{Map}(G, \mathbb{C})_+ = \bigoplus_{k=1}^m \mathbb{C}[k]$; if $G = S^1$, first take the space of maps $S^1 \to \mathbb{C}$ which extend holomorphically over the unit disc and then within that take the S^1-equivariant complement to the constant maps. This is a completion of the direct sum $\bigoplus_{k\geq 1} \mathbb{C}[k]$.

In the case that $G = C_p$ with $p = 2m+1$, the top exterior power of $TM \otimes \operatorname{Map}(G,\mathbb{C})_+$ is the trivial line bundle $\mathbb{C}[dm(m+1)/2]$. For $G = S^1$, this is the term which is renormalised to $\mathbb{C}[-d/12]$.

Since $\operatorname{Map}(G,\mathbb{C})_+$ decomposes as the sum $\bigoplus \mathbb{C}[k]$, the bundle $TM \otimes \operatorname{Map}(G,\mathbb{C})_+$ similarly decomposes. Thus:

$$\operatorname{ch}_\xi \Lambda_{-1} TM \otimes \operatorname{Map}(G,\mathbb{C})_+ = \operatorname{ch}\left(\bigotimes \Lambda_{-\xi^k} T_{\mathbb{C}} M\right),$$

where the indexing set of the product is $1 \leq k \leq m$ for $G = C_p$ with $p = 2m+1$, and is $1 \leq k < \infty$ for $G = S^1$.

Substituting this into the expression in theorem 2.6 yields Witten's formula for $G = S^1$ and a proof of theorem 1.2 for $G = C_p$:

Theorem 3.4. *The Witten genus, as the formal index of the Dirac operator on* $\operatorname{Map}(S^1, M)$*, is:*

$$(-1)^{d/2} \xi^{-d/24} \left\langle \widehat{\mathcal{A}}(TM) \operatorname{ch}\left(\bigotimes_{k>0} (\Lambda_{-\xi^k} T_{\mathbb{C}} M)^{-1}\right), [M] \right\rangle,$$

and the truncated Witten genus, as the index of the Dirac operator on $\operatorname{Map}(C_p, M)$*, is:*

$$(-1)^{(m+1)d/2} \xi^{dm(m+1)/4} \left\langle \widehat{\mathcal{A}}(TM) \operatorname{ch}\left(\bigotimes_{k=1}^{m} (\Lambda_{-\xi^k} T_{\mathbb{C}} M)^{-1}\right), [M] \right\rangle.$$

4 The Loop Space of Projective Space

In this section we explain briefly the connection between the family of finite dimensional projective spaces, $\{\mathbb{P}\operatorname{Map}(C_p, \mathbb{C}^{n+1})\}$, and the loop space $\operatorname{Map}(S^1, \mathbb{CP}^n)$.

In the following, for a complex vector space V, V^* will denote the non-zero vectors in V. For a subset $U \subseteq V^*$ which is closed under the action of the non-zero complex numbers, we shall write $\mathbb{P}U$ for the quotient $U/\mathbb{C}^*$.

Let n, p be positive integers. Let G be either S^1 or C_p. We start with the fibration $\mathbb{C}^* \to \mathbb{C}^{n+1,*} \to \mathbb{CP}^n$. This defines a fibration of mapping spaces:

$$\operatorname{Map}(G, \mathbb{C}^*) \to \operatorname{Map}(G, \mathbb{C}^{n+1,*}) \to \operatorname{Map}(G, \mathbb{CP}^n).$$

The space $\operatorname{Map}(G, \mathbb{C}^{n+1,*})$ consists of maps from G into $\mathbb{C}^{n+1}$ which are never zero. It thus includes in the space $\operatorname{Map}(G, \mathbb{C}^{n+1})^*$ of maps from G into $\mathbb{C}^{n+1}$ which are not always zero. This inclusion descends to the projective spaces. The projection $\operatorname{Map}(G, \mathbb{C}^{n+1,*}) \to \operatorname{Map}(G, \mathbb{CP}^n)$ factors through the projective space $\mathbb{P}\operatorname{Map}(G, \mathbb{C}^{n+1,*})$. Thus there is a diagram in which all maps are G-equivariant:

$$\begin{tikzcd}
\mathbb{P}\,\mathrm{Map}(G, \mathbb{C}^{n+1,*}) \arrow[r] \arrow[d] & \mathbb{P}\,\mathrm{Map}(G, \mathbb{C}^{n+1}) \\
\mathrm{Map}(G, \mathbb{CP}^n) &
\end{tikzcd} \tag{3}$$

Restriction to the subgroup $C_p \subseteq S^1$ defines a map from this diagram for S^1 to the diagram for C_p. The map $\mathrm{Map}(S^1, \mathbb{CP}^n) \to \mathrm{Map}(C_p, \mathbb{CP}^n)$ is part of the functor from loop spaces to cocyclic objects applied to projective space.

Conversely, the spaces $\mathbb{P}\,\mathrm{Map}(C_p, \mathbb{C}^{n+1})$ can be used to define a filtration of $\mathbb{P}\,\mathrm{Map}(S^1, \mathbb{C}^{n+1})$. Let $a \in \mathbb{Z}$ and let $b = a + p - 1$. Let $\mathbb{C}^{n+1}[z^{-1}, z]_a^b$ denote the space of Laurent polynomials in z with coefficients in $\mathbb{C}^{n+1}$ of the form $v_a z^a + \cdots + v_b z^b$. The circle acts on this space via $\zeta \cdot z^k \to \zeta^k z^k$ and this action descends to the projective space, $\mathbb{PC}^{n+1}[z^{-1}, z]_a^b$. This projective space includes S^1-equivariantly in $\mathbb{P}\,\mathrm{Map}(S^1, \mathbb{C}^{n+1})$.

The space $\mathbb{C}^{n+1}[z^{-1}, z]_a^b$ can be C_p-equivariantly identified with the space of maps into $\mathbb{C}^{n+1}$, $\mathrm{Map}(C_p, \mathbb{C}^{n+1})$. This identification takes a polynomial in and restricts it to the subgroup $C_p \subseteq S^1$. That this is an identification follows from the fact that the space of polynomials of degree $p - 1$ in a complex vector space V is identified with V^p by evaluating at p distinct points in $\mathbb{C}$.

Thus a choice of $a \in \mathbb{Z}$ defines an extension to the circle of the C_p-action on $\mathbb{P}\,\mathrm{Map}(C_p, \mathbb{C}^{n+1})$ and defines an S^1-equivariant map $\mathbb{P}\,\mathrm{Map}(C_p, \mathbb{C}^{n+1}) \to \mathbb{P}\,\mathrm{Map}(S^1, \mathbb{C}^{n+1})$. The family of such maps taken over $a \in \mathbb{Z}$ and $p \in \mathbb{N}$ defines a filtration of $\mathbb{P}\,\mathrm{Map}(S^1, \mathbb{C}^{n+1})$.

5 The Index of the Dirac Operator

In this section we shall calculate the index of the equivariant Dirac operator on $\mathbb{P}\,\mathrm{Map}(G, \mathbb{C}^{n+1})$. First we need to show that in the case of G of finite odd order, the Dirac operator exists and is equivariant under the group action.

Proposition 5.1. *In the case that G has finite odd order, if* $\mathbb{CP}^n$ *is spin then* $\mathbb{P}\,\mathrm{Map}(G, \mathbb{C}^{n+1})$ *is a spin manifold and the action of G lifts to the spin structure.*

Proof. Let p be the order of G. The manifold $\mathbb{P}\,\mathrm{Map}(G, \mathbb{C}^{n+1})$ is non-equivariantly $\mathbb{CP}^{p(n+1)-1}$. Now, a projective space $\mathbb{CP}^k$ is spin if and only if k is odd. Thus if $\mathbb{CP}^n$ is spin, n is odd and so $\mathbb{P}\,\mathrm{Map}(G, \mathbb{C}^{n+1})$ is also spin.

Because $\mathbb{P}\,\mathrm{Map}(G, \mathbb{C}^{n+1})$ is a complex manifold, proposition 2.2 says that the group action lifts to the spin structure if and only if a certain line bundle can be *equivariantly* square rooted. We already know that it can be non-equivariantly square rooted.

Let γ be the canonical line bundle over $\mathbb{P}\,\mathrm{Map}(G, \mathbb{C}^{n+1})$. This has a natural action of G. Since any line bundle over $\mathbb{P}\,\mathrm{Map}(G, \mathbb{C}^{n+1})$ is isomorphic to a tensor power of γ or its dual, any line bundle has an action of G. If L is a line bundle with a G-action which is non-equivariantly isomorphic to $\gamma^{\otimes 2k}$ then there is some r such that equivariantly $L = \gamma^{\otimes 2k} \otimes \mathbb{C}[r]$. As G is odd, there is some s such that $\mathbb{C}[s] \otimes \mathbb{C}[s] = \mathbb{C}[r]$. Let $L' = \gamma^{\otimes k} \otimes \mathbb{C}[s]$, then $L' \otimes L' = L$ equivariantly. Thus any line bundle which can be non-equivariantly square rooted can be equivariantly square rooted. □

We wish to apply the Atiyah-Singer fixed point theorem to the manifold $\mathbb{P}\operatorname{Map}(G, \mathbb{C}^{n+1})$. To do this we need to determine the fixed point set and the normal bundle to its inclusion.

Proposition 5.2. *The fixed point set of* $\mathbb{P}\operatorname{Map}(G, \mathbb{C}^{n+1})$ *is a number of disjoint copies of* $\mathbb{CP}^n$ *indexed by the (smooth) homomorphisms* $G \to S^1$.

Proof. Let $\alpha : G \to \mathbb{C}^{n+1}$ represent a point in the fixed point set. As it is a fixed point, for $\zeta \in G$ the map $\nu \to \alpha(\nu\zeta)$ represents the same point as α. Therefore there is some $\lambda_\zeta \in \mathbb{C}^*$ such that $\alpha(\nu\zeta) = \lambda_\zeta \alpha(\nu)$ for all $\nu \in G$. In particular, $\alpha(\zeta) = \lambda_\zeta \alpha(1)$ and so α is determined by $\alpha(1)$ and the assignment $\zeta \to \lambda_\zeta$.

It is easy to see that for $\zeta, \nu \in G$ then $\lambda_{\zeta\nu} = \lambda_\zeta \lambda_\nu$ whence the assignment $\zeta \to \lambda_\zeta$ is a group homomorphism $G \to \mathbb{C}^*$, smooth if G is the circle. As G is compact, this must in fact be a homomorphism into S^1.

If α and β represent the same element then there is some $\mu \in \mathbb{C}^*$ such that $\alpha = \mu\beta$ from which it is simple to deduce that the group homomorphisms must be the same and that $\alpha(1)$ and $\beta(1)$ represent the same element in $\mathbb{CP}^n$.

Thus the fixed point set is $\mathbb{CP}^n \times \hom(G, S^1)$, the inclusion given by $([z], h) \to [\zeta \to h(\zeta)z]$. Note that $\hom(G, S^1)$ is a discrete set and so the fixed point set is a number of copies of $\mathbb{CP}^n$. □

Proposition 5.3. *The normal bundle to the inclusion of one component of the fixed point set is* $\widetilde{N} = N \oplus (\operatorname{Map}(G, \mathbb{C})_0)_\mathbb{R}$ *where* N *denotes the normal bundle to the inclusion of* $\mathbb{CP}^n$ *in* $\operatorname{Map}(G, \mathbb{CP}^n)$ *as the fixed point set and* $\operatorname{Map}(G, \mathbb{C})_0$ *denotes the orthogonal complement of the constant loops in* $\operatorname{Map}(G, \mathbb{C})$.

Proof. It is clear that if α represents an element of the fixed point set in $\mathbb{P}\operatorname{Map}(G, \mathbb{C}^{n+1})$ then α is a never-zero map and therefore also lies in the subspace $\mathbb{P}\operatorname{Map}(G, \mathbb{C}^{n+1,*})$. Since this space is open in $\mathbb{P}\operatorname{Map}(G, \mathbb{C}^{n+1})$, the normal bundles to the inclusions of the fixed point sets are the same.

Consider the inclusion in $\mathbb{P}\operatorname{Map}(G, \mathbb{C}^{n+1,*})$ of one component of the fixed point set. Under the projection to $\operatorname{Map}(G, \mathbb{CP}^n)$ this component is taken onto the fixed point set in $\operatorname{Map}(G, \mathbb{CP}^n)$. Therefore the normal bundle is the sum of the normal bundle to the inclusion of the fixed points in $\operatorname{Map}(G, \mathbb{CP}^n)$ and the vertical tangent bundle of $\mathbb{P}\operatorname{Map}(G, \mathbb{C}^{n+1,*})$. The fibre of this projection is $\operatorname{Map}(G, \mathbb{C}^*)/\mathbb{C}^*$ with the quotient by the subgroup of constant loops. Therefore the vertical tangent bundle is $(\operatorname{Map}(G, \mathbb{C})_0)_\mathbb{R}$. □

Recall that N is the underlying real bundle of a complex bundle with vanishing first Chern class, call this bundle U. Therefore $\widetilde{N}$ is the underlying real bundle of $\widetilde{U} := U \oplus \operatorname{Map}(G, \mathbb{C})_0$ which also has vanishing first Chern class. Thus we can use theorem 2.6 to compute the equivariant index of the Dirac operator. As in section 3 we shall consider the case where G is not of even order.

To do this, we need to determine the group action on the top exterior power of $\widetilde{U}$ and to compute $\operatorname{ch}_\xi \Lambda_{-1}\widetilde{U}$. In fact, we only need to compute the correction terms to these coming from the $\operatorname{Map}(G, \mathbb{C})_0$ factor as the U factor will yield the term in the Witten genus or truncated Witten genus as appropriate.

If G is finite, so that the expression makes sense, the top exterior power of $\operatorname{Map}(G, \mathbb{C})_0$ is equivariantly trivial, therefore the top exterior powers of $\widetilde{U}$ and U are equivariantly the same. We formally set this to be so also in the case of the circle.

Then $\operatorname{ch}_\xi \Lambda_{-1}\widetilde{U} = (\operatorname{ch}_\xi \Lambda_{-1}U)(\operatorname{ch}_\xi \Lambda_{-1} \operatorname{Map}(G, \mathbb{C})_0)$ so we need to compute the second term in this product. For G finite, of odd order $p = 2m + 1$, $\operatorname{Map}(G, \mathbb{C})_0$ decomposes as the sum of $\mathbb{C}[k]$ for $1 \leq |k| \leq m$. Thus $\operatorname{ch}_\xi \Lambda_{-1} \operatorname{Map}(G, \mathbb{C})_0$ is $\prod_{k=1}^m (1-\xi^k)(1-\xi^{-k})$ which can be rearranged to $(-1)^m \xi^{-m(m+1)/2} \prod_{k=1}^m (1-\xi^k)^2$. In the infinite case, we renormalise this to $\xi^{1/12} \prod_{k\geq 1}(1 - \xi^k)^2$.

As this term contains no topological information, it can be taken to the front of the formula in theorem 2.6. Thus the contribution from one component of the fixed point set to the equivariant index of the Dirac operator on $\mathbb{P}\operatorname{Map}(G, \mathbb{C}^{n+1})$ is, for $G = S^1$ formally:

$$\xi^{-1/12} \left(\prod_{k>0} (1 - \xi^k)^{-2} \right) W(M)(\xi) \tag{4}$$

and for $G = C_p$:

$$(-1)^m \xi^{m(m+1)/2} \left(\prod_{k=1}^{m} (1 - \xi^k)^{-2} \right) W_p(M).$$

which is a truncation of equation (4): the $\xi^{-1/12}$ truncates to $\xi^{m(m+1)/2}$, the infinite product $\prod_{k>0}(1 - \xi^k)^{-2}$ truncates to $\prod_{k=1}^m (1 - \xi^k)^{-2}$, and the Witten genus truncates to the truncated Witten genus.

Since the components of the fixed point set in $\mathbb{P}\operatorname{Map}(G, \mathbb{C}^{n+1})$ are the same and have the same normal bundle, the contribution to the index of the Dirac operator on $\mathbb{P}\operatorname{Map}(G, \mathbb{C}^{n+1})$ is the same from each. When $G = C_p$ there are p components which yields the formula in theorem 1.3. When $G = S^1$, there are a countably infinite number of components and so the summation is purely formal. The expression in theorem 1.3 is the truncation of this formal sum.

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