Semi-Infinite de Rham Theory
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Andrew Stacey

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Abstract

I shall start by explaining just what "semi-infinite" means and how it gives rise to interesting structures in geometry. These structures are studied as part of String Theory and are closely related to Floer Theory. Manifolds which carry a "semi-infinite" structure include loop spaces ($C^\infty(S^1, M)$ for M a closed manifold) and the theory seems to have particular simplicities when the original manifold is symplectic.

In time, it is hoped to develop many semi-infinite variants of "ordinary" geometrical objects such as K-Theory and Index Theory. For the moment, the main area of research appears to be into semi-infinite cohomology. Floer theory can be thought of as a "semi-infinite" variant of Morse theory. There is also a "semi-infinite" Lie algebra theory put forward by Feigin and Frenkel amongst others.

I shall outline how one can construct a semi-infinite de Rham theory. Along the way, I hope to illustrate how one deals in general with semi-infinite objects (by "taming" them). I shall also give some indications of where elements of the construction can be used as starting points for considering semi-infinite K-Theory and Index Theory, and how the natural circle action on the loop space can be factored in.

Contents

1 **Semi-What?** 2

2 **Relevance of Semi-Infinite Theories** 3

3 **The Construction of $\Lambda_{\mathbf{si}} E^*$** 4

4 **Taming the Infinite** 5

5 **Further Restrictions May Apply** 7

6 **To Infinity and Beyond** 7

1 Semi-What?

It is usual to start a talk on a subject with a brief overview of the topic, perhaps a road map of the seminar, and connections with other areas. Before I do all that, I would like to explain the term "semi-infinite" that occurs in the title. Until I have done that, it's hard to talk about anything else.

The term "semi-infinite" is not as scary as it appears. It is merely a method of choosing a "half-way" point between zero and infinity. There is a very simple illustration of this point from set theory. We know that the negative integers and all the integers are "the same size" and we all know how to construct a bijection between them. However, if we include the negative integers in all integers in the natural way, we find that we still have a lot left over. If we regard the choice of a subset of $\mathbb{Z}$ as meaning that we have "counted" all the elements of that subset then the choice of $-\mathbb{N}$ in $\mathbb{Z}$ means that we have "counted" all the negative integers[1]. We have "counted" a lot of numbers - an infinite number - but we still have a lot left - also infinite. Thus the inclusion of $-\mathbb{N}$ in $\mathbb{Z}$ denotes that we have "counted" "half" the numbers of $\mathbb{Z}$. If we only allow subsets of the form $(-\infty, n]$, we can mark the subset of "counted" elements by the point $n + 1$ on the number line. Thus $-\mathbb{N}$ corresponds to 0 and so when we say that we are at zero, we have actually already "counted" all the negative numbers.

Anyone with a Quantum Physics background should at this point be thinking of the Dirac sea. In that model, an electron is not one particle, it is one particle more than we had already and the number that we had already is infinite. Although mathematically we would argue that there is "no difference" between $-\mathbb{N}$ and $-\mathbb{N} \cup \{0\}$, Physicists say that there is because they can build some apparatus that will detect it.

These two examples give the essence of a semi-infinite object. The last concept we need is that of the difference between two such objects. The need for this is clear from the Dirac sea model. The only situations we will actually observe are with a finite number of electrons and a finite number of positrons

[1]we shall use the notation $\mathbb{N}$ to denote $\{1, 2, \dots\}$ and $\mathbb{N}_0$ for $\{0\} \cup \mathbb{N}$.

(which in this model are considered as "missing" electrons). Translating this to the integer model, we may well observe that instead of the pair $(\mathbb{Z}, -\mathbb{N})$ we actually have $(\mathbb{Z}, \{\ldots, -3, 0, 10, 121\})$ but we will never observe the pair $(\mathbb{Z}, -2\mathbb{N})$ or $(\mathbb{Z}, \mathbb{N})$ as these would involve extracting and or adding an infinite number of numbers which would, back in the Dirac sea model, take an infinite amount of energy.

Rather than sets or electrons, what we are actually concerned with is vector spaces as these lead us into geometry as the model spaces of manifolds. The definition of a *polarised vector space* is:

Definition 1.1 *A* polarisation *on an infinite dimensional, locally convex, topological vector space E is a class of decompositions $E \cong E_- \oplus E_+$ where $E_\pm$ are infinite dimensional, closed subspaces of E. Two such decompositions $E_- \oplus E_+$ and $E'_- \oplus E'_+$ are equivalent if the natural maps $E_\pm \to E'_\pm$ are Fredholm and $E_\pm \to E'_\mp$ are compact.*

A polarised vector space *is a vector space together with a polarisation.*

If the word "compact" were replaced by the words "finite rank", the equivalence condition would be the same as saying that the subspace $E_- \cap E'_-$ is of finite codimension in both E_- and E'_- with a similar condition on the positive spaces. For topological reasons, "finite rank" isn't enough to be going with and "compact" is the general alternative. With extra structure on E, say a Hilbert space structure, the word "compact" can be replaced by other things; in the case of a Hilbert space the chosen phrase is "Hilbert-Schmidt".

Clearly, a vector space with structure leads to consideration of manifolds with that structure:

Definition 1.2 *A* polarised manifold *is a manifold M modelled on a polarised vector space such that the structure group of M preserves the polarisation.*

In other words, the structure group of M acts on E in such a way that one choice of decomposition, $E_- \oplus E_+$ is mapped to an equivalent one. The full group of such transformations is denoted $\mathrm{Gl}_{\mathrm{res}}(E)$, however this may not have the structure of a regular Lie group and this leads to the phraseology of the above definition.

2 Relevance of Semi-Infinite Theories

Now that we are happy with the concept of "semi-infinite" and have an idea as to what a polarised manifold is, we can discuss the relevance of the subject. The initial questions that arise are:

1. Are there any polarised manifolds, and if so are there any interesting ones?
2. Given a polarised manifold, what can you do with it?

There are polarised manifolds. In fact, all loop spaces of almost complex finite dimensional manifolds are polarised manifolds[2]. These could be classed as the "most interesting" polarised manifolds. Another class is obtained by choosing a separable, polarised vector space E and then looking at manifolds modelled on that space, for example $\mathbb{P}(E)$ or $\mathrm{Gr}_k(E)$. These could be classed as the "most tractable" polarised manifolds.

The model space for the loop space of an almost complex manifold is the nuclear Fréchet space $L\mathbb{C}^n = C^\infty(S^1, \mathbb{C}^n)$. This is polarised in the following way: consider the differential operator $-i\frac{d}{dt}$ and consider its positive and negative eigenvectors, usually denoted z_i^k for $k \in \mathbb{Z}$, $1 \leq i \leq n$. The polarisation is the equivalence class of the closure of the span of $\{z_i^k : k \geq 0\}$, denoted $L_+\mathbb{C}^n$.

By a "semi-infinite theory" we mean any theory which uses the polarisation information in a non-trivial way. To see the relevance of such a theory, recall that string theory says that there aren't any particles around, just strings vibrating away in the ether. The vibrations on these strings have a direction and it just so happens that this direction coincides with the polarisation of the tangent bundle at that loop. Thus any theory which is designed to detect the direction of the loop must involve using the polarisation of the loop space.

In mathematics, so far the main semi-infinite theory is Floer theory. This is a variant of Morse theory which uses the polarisation. There is also a Lie algebra theory which has resemblances to finite dimensional Lie algebra cohomology. Apart from just extending this list to include de Rham theory, the techniques involved in constructing the theory should be extendible to other constructions which use the de Rham complex, such as index theory.

As an example, consider the signature operator. In finite dimensions all the interesting information is contained in the middle cohomology group. Thus to discuss the signature of the loop space, one needs to know what the middle cohomology group is and this is where the semi-infinite stuff comes in.

Thus the two aims of a semi-infinite de Rham theory are firstly to provide a "choice-invariant" semi-infinite cohomology theory (as opposed to Floer theory which depends upon a choice of Floer function) and secondly to provide a place from which to consider more exotic constructions.

3 The Construction of $\Lambda_{\mathbf{si}} E^*$

Essentially, a cohomology theory has two ingredients: the cochain groups and the differential. In this section, we take a brief look at how to construct the cochain groups.

By analogy with ordinary de Rham theory, the cochain groups will look something like $\Gamma(M, \Lambda^k_{\mathrm{si}} T^* M)$ for M modelled on E. Thus the key is the construction of $\Lambda^k_{\mathrm{si}} E^*$. The idea uses the following theorem, see for example [PS86], proposition 2.9.2:

[2]With a slight alteration to the definition of a polarisation - equivalent in the complex case - all loop spaces of finite dimensional manifolds are polarised

Theorem 3.1 *Let V be a complex finite dimensional vector space. Let* Det $\to$ $\mathrm{Gr}_k(V)$ *be the determinant bundle over the space of k planes in V. Then* $\Lambda^k V^* \cong \Gamma_{hol}(\mathrm{Gr}_k(V), \mathrm{Det}^*)$.

Thus to construct $\Lambda^k_{\mathrm{si}} E^*$ for a polarised vector space E, we look for a convenient Grassmannian with a determinant bundle over it. Fortunately, one just happens to be lying around:

Definition 3.2 *The* restricted Grassmannian *of a complex, polarised vector space E is the space of all positive parts of representatives of the polarisation of E: $W \in \mathrm{Gr}_{res}(E)$ if and only if:*

1. $p_+ : W \to E_+$ *is Fredholm, and*
2. $p_- : W \to E_-$ *is compact.*

Again, compact may and will be replaced by Hilbert-Schmidt in the case of a Hilbert space.

There is a determinant bundle over this space and it is a complex line bundle over a complex manifold, hence the space $\Gamma_{\mathrm{hol}}(\mathrm{Gr}_{\mathrm{res}}, \mathrm{Det}^*)$ makes sense. To get the vector space $\Lambda_{\mathrm{si}} T^* M$ over M, we thus construct the restricted Grassmannian at each point, over that construct the determinant bundle and on fibres, take holomorphic sections. Thus:

$$\Lambda_{\mathrm{si}} T^* M := \Gamma_{\mathrm{hol}}(\mathrm{Gr}_{\mathrm{res}}(TM), \mathrm{Det}^*)$$

The restricted Grassmannian carries a grading, given by the index of the Fredholm operator $p_+ : W \to E_+$. However, because this requires a choice of E_+, it is not a natural grading. Over a manifold, therefore, this grading may not be absolutely definable. The problem being that if one goes around a loop in M, one may end up in a different component of $\mathrm{Gr}_{\mathrm{res}}(TM)$ to the one started in. This defines a *periodicity*, $b_1(M) \in H^1(M, \mathbb{Z})$ and the grading is well-defined up to that periodicity.

Anyone familiar with Floer theory will be relieved to know that the periodicity of the loop space of an almost complex, simply connected manifold is that predicted by Floer theory, namely double the transgression of the first Chern class.

Naïvely, one now puts $C^k_{\mathrm{si}}(M) = \Gamma(M, \Lambda^k_{\mathrm{si}} T^* M)$ with appropriate periodicity on the gradings. This naïve definition has problems, which we will encounter when we try to define the differential.

4 Taming the Infinite

An essential part of semi-infinite theory is keeping things relatively finite. We saw this in the definition of a polarised vector space, although the subspaces were allowed to be infinite dimensional, the differences had to be (almost) finite. We encounter the same problem when it comes to defining the differential.

In finite dimensions, the de Rham differential is defined by the composition in local coordinates for $U \subseteq V$ open:

$$\Gamma(U, \Lambda^k V^*) \xrightarrow{D} \Gamma(U, V^* \otimes \Lambda^k V^*) \xrightarrow{\wedge} \Gamma(U, \Lambda^{k+1} V^*)$$

Using the isomorphism $\Lambda^k V^* \cong \Gamma_{\mathrm{hol}}(\mathrm{Gr}_k(V), \mathrm{Det}^*)$, we can find the right formulation for the wedge product in terms of the Grassmannian model and this formulation extends to the semi-infinite case. Thus the right hand side of the above makes sense in the semi-infinite case. However, the left hand side does not.

The differential map D is not really as stated above. Its image is actually the space $\Gamma(U, \hom(V, \Lambda^k V^*))$. In finite dimensions, this is the same as the space in the middle above. In infinite dimensions, it is a lot bigger.

To really understand this requires some functional analysis, but the idea is that the wedge map is like a trace map and therefore can only be defined on operators which have a trace. In finite dimensions, all operators have a trace. In infinite dimensions, only a select few do so. For example, the identity map does not have a trace. When dealing with smooth loop spaces, the model space is a nuclear Fréchet space and the phrase "a select few" is more accurately translated as "almost but not quite all". Thus locally we have:

$$\begin{array}{ccccc} \Gamma(U, \Lambda^k_{\mathrm{si}} T^*U) & \xrightarrow{D} & \Gamma(U, \hom(TU, \Lambda^k_{\mathrm{si}} T^*U)) & & \\ & & \uparrow & & \\ & & \Gamma(U, T^*U \otimes \Lambda^k_{\mathrm{si}} T^*U) & \xrightarrow{\wedge} & \Gamma(U, \Lambda^{k+1}_{\mathrm{si}} T^*U) \end{array}$$

This suggests as an initial definition, $C^k(U) = \{s \in \Gamma(U, \Lambda^k_{\mathrm{si}} T^*U) : Ds \in \Gamma(U, T^*U \otimes \Lambda^k_{\mathrm{si}} T^*U)\}$, i.e. those sections with trace-class derivative. The problem with this is that if $s \in C^k(U)$ there is nothing to guarantee that $ds = \wedge Ds \in C^k(U)$. There are two ways around this, but they turn out to be equivalent. The first is to take $\Omega^k_{\mathrm{si}}(U) = d^{-1} C^{k+1}(U)$, the other is to take $\Omega^k_{\mathrm{si}}(U) = C^k(U) + dC^{k-1}(U)$. That both form cochain complexes with differential d is due to the fact that $d^2 = 0$.

Thus we have a complex $\Omega^k_{\mathrm{si}}(U)$ defined for $U \subseteq E$ open. There are two problems with extending this to a manifold modelled on E. Firstly, if $\phi : U \to V$ is a diffeomorphism, there is no reason to suppose that if $s \in \Omega^k_{\mathrm{si}}(V)$ then $\phi^* s \in \Omega^k_{\mathrm{si}}(U)$. The change of variables formula has an extra term which may not be trace class. Secondly, I have glossed over one aspect of the construction: namely the group action on $\Lambda_{\mathrm{si}} E^*$. In general, given a Lie group G acting on E in such a way as to preserve the polarisation, G acts *projectively* on $\Lambda_{\mathrm{si}} E^*$. Thus one has to work with an extension of G and this may add further terms to complicate the change of variables formulae.

My initial way around this was to observe that for loop spaces, these effects cancel each other out. Namely, no diffeomorphism with differential in $L\mathrm{Gl}_n$ adds a trace class component. Thus one defines local forms in the following way: a local form consists of a chart U together with a section supported in U

such that the differential is trace-class. Because no diffeomorphism adds a trace class component, the choice of U is essentially unique so we don't have to worry about changing variables. We then take the linear span of such sections. That such sections exist is easily shown using partitions of unity.

5 Further Restrictions May Apply

It is one thing to define a theory, another to calculate it. There is a theorem which states that any Hilbert manifold is diffeomorphic to itself crossed with $\mathbb{R}$. With some extra assumptions on the diffeomorphism, we get an isomorphism $H^k_{\mathrm{si}}(M) \cong H^{k+1}_{\mathrm{si}}(M)$ for all k and also some strong evidence that $H^k_{\mathrm{si}}(M) = 0$ for all k. This is not the answer we want.

One route is to find a refinement of the theory which allows for a Thom isomorphism. This can be done for Hilbert manifolds and using this refinement we get the nice result that:

$$H^k_{\mathrm{si}}(\mathbb{P}H) = \begin{cases} \mathbb{C} & \text{if } k \in 2\mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

The problem with this is that it uses an integration theory which means that it only applies to Banach (including Hilbert) manifolds. To find a refinement that applies more generally, we need to look for additional structure.

One piece of extra structure that is often associated with polarisations is a filtration. This is a sequence, infinite at both ends, of objects which limit to the polarised object. The simplest case is that of $\mathbb{P}H$. Consider H to have a basis indexed by $\mathbb{Z}$ with obvious positive and negative parts. Consider $\mathbb{CP}^b_a \subseteq \mathbb{P}H$ consisting of the projective space of only the span of the basis elements $\{\zeta_a, \dots, \zeta_b\}$. In cohomology we have the natural pull-back map $H^k(\mathbb{CP}^{b+1}_a) \to H^k(\mathbb{CP}^b_a)$ together with the push-forward Thom-Pontrijagin map $H^k(\mathbb{CP}^b_a) \to H^{k+2}(\mathbb{CP}^b_{a-1})$. The double limit of this system is obviously the answer given above. Therefore a refinement of semi-infinite cohomology in which the limit of a filtration gives the right answer would be a good one to use. For Hilbert manifolds, again such a refinement exists.

6 To Infinity and Beyond

For loop spaces, the first stage of the quest is thus an appropriate filtration. The key is the Thom-Pontrijagin map. This is a theme that has been played at various times for the last few years, most notably by Cohen, Jones, and Segal [CJS95], [CJ02], and [Coh01]. What one looks for is a sequence of Thom spaces:

$$X^{-\zeta_1} \leftarrow X^{-\zeta_2} \leftarrow X^{-\zeta_3} \leftarrow \cdots$$

Each of the ζ_i may be infinite dimensional, but the quotients ζ_i/ζ_{i+1} are finite dimensional, allowing the definition of a Thom-Pontrijagin map. In the case of a loop space LM, Cohen has proposed a sequence where each quotient is isomorphic to the pullback of TM under an evaluation map. The problem with this is that the Thom-Pontrijagin maps are multiplication by the Euler class of the bundle, which in the case of this sequence is nilpotent.

Moving into the arena of hazy speculation, if one changes theory slightly to take into account the natural circle action and defines an equivariant theory, these Euler classes become invertible and hence one should get a reasonably behaved theory.

In fact, what we are searching for is a marriage between my theory and string topology in an equivariant setting which should then define a full-blown topological quantum field theory.

One reason that the circle action should find a natural place in semi-infinite theory is the following: consider the tangent bundle to the loop space over the constant loops (note that these are the fixed points of the circle action). Although an infinite dimensional vector bundle, it splits very nicely as a sum of finite dimensional representations of the circle, each of which is isomorphic to the tangent bundle of the original space. Moreover, the semi-infinite exterior power has the same property: there is a circle action and each factor in the corresponding decomposition is finite. Therefore what starts out as merely infinite dimensional becomes an infinite number of the same finite dimensional object. This yields a method of "taming the infinite" compatible with those already seen.

References

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