

FINITELY PRESENTED FUNCTORS AND DEGENERATIONS

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Dedicated to Claus M. Ringel on the occasion of his 60th birthday

ABSTRACT. Given two d -dimensional Λ -modules M and N , then M degenerates to N if and only if there exists an exact sequence of the form $0 \rightarrow U \rightarrow U \oplus M \rightarrow N \rightarrow 0$ for some $U \in \text{mod } \Lambda$ [6]. Having this as a starting point, we give in this paper a characterization of degenerations by the existence of a certain finitely presented functor. This gives new information about U from the sequence above. We show how this new information can be used to prove that $M \not\leq_{deg} N$ even when $M \oplus X \leq_{deg} N \oplus X$ for some X for modules over $\Lambda_q = k\langle x, y \rangle / \langle x^2, y^2, xy + qyx \rangle$, $q \neq 0$ in k , where k is an algebraically closed field.

1. INTRODUCTION

Let k be an algebraically closed field and Λ a finitely generated associative k -algebra with unit. We will denote by $\text{mod } \Lambda$ the category of finite-dimensional left Λ -modules.

A d -dimensional (left) Λ -module M is a d -dimensional vector space M together with a multiplication by Λ from the left. By choosing a basis in M one can identify M with the vector space k^d . If $\lambda_1, \lambda_2, \dots, \lambda_l$ is a generating set for Λ over k , then the multiplication of M by λ_i induces a k -endomorphism of k^d which can be represented by a $d \times d$ -matrix over k . Thus, M corresponds to a unique l -tuple $m = (m_1, m_2, \dots, m_l)$ of $d \times d$ -matrices over k such that for all polynomials f in l non-commuting variables over k with the property that $f(\lambda_1, \lambda_2, \dots, \lambda_l) = 0$ in Λ , we have $f(m_1, m_2, \dots, m_l) = 0$ in the ring of $d \times d$ -matrices over k . Conversely, each such l -tuple m corresponds to a ring homomorphism $\phi_m : \Lambda \rightarrow \text{End}_k(k^d)$ and hence gives a module structure M on k^d in the obvious way. We denote by $\text{mod}_\Lambda^d(k)$ the set of all such l -tuples. It is an affine variety.

The general linear group $\text{Gl}_d(k)$ acts on $\text{mod}_\Lambda^d(k)$ by conjugation, $g * x = (gx_1g^{-1}, \dots, gx_lg^{-1})$ for $g \in \text{Gl}_d(k)$, $x \in \text{mod}_\Lambda^d(k)$, and the orbits correspond to the isomorphism classes of d -dimensional Λ -modules (see [3]). We denote by $\mathcal{O}(m)$ the $\text{Gl}_d(k)$ -orbit of $m \in \text{mod}_\Lambda^d(k)$. The dimension of $\mathcal{O}(m)$ is given by

$$\dim \mathcal{O}(m) = \dim \text{Gl}_d(k) - \dim \text{Aut}_\Lambda(M) = d^2 - \dim \text{End}_\Lambda(M).$$

Let M and N be two d -dimensional left Λ -modules corresponding to m and n in $\text{mod}_\Lambda^d(k)$ respectively. We say that M degenerates to N if n belongs to the Zariski closure $\overline{\mathcal{O}(m)}$ of $\mathcal{O}(m)$, and we denote this by $M \leq_{deg} N$ (see [2]).

Clearly, \leq_{deg} is a partial order on the set of isomorphism classes of d -dimensional modules.

Riedtmann showed in [5] that if Λ is a finite dimensional algebra and if there is an exact sequence of the form

$$(1) \quad 0 \longrightarrow U \longrightarrow U \oplus M \longrightarrow N \longrightarrow 0$$

for some $U \in \text{mod } \Lambda$, or of the form

$$(2) \quad 0 \longrightarrow N \longrightarrow M \oplus V \longrightarrow V \longrightarrow 0$$

for some $V \in \text{mod } \Lambda$, then M degenerates to N . In particular, if

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an exact sequence, then $B \leq_{deg} A \oplus C$.

Zwara showed in [6] that for finite dimensional algebras the existence of a sequence of the form (1) is equivalent to the existence of a sequence of the form (2), and he showed in [7] that this is equivalent to $M \leq_{deg} N$.

It is easy to see that this characterization also holds for finitely generated algebras: If Λ is a finitely generated algebra and M a d -dimensional Λ -module, then $\Lambda/\text{ann}(M)$ is a finite dimensional algebra. Note that

$$\mathcal{O}(m) \subseteq \{x \in \text{mod}_{\Lambda}^d(k) \mid \text{ann}(M) \cdot x = 0\} = \text{mod}_{\Lambda/\text{ann}(M)}^d(k).$$

Since $\text{mod}_{\Lambda/\text{ann}(M)}^d(k)$ is closed in $\text{mod}_{\Lambda}^d(k)$, it follows that $\overline{\mathcal{O}(m)} \subseteq \text{mod}_{\Lambda/\text{ann}(M)}^d(k)$. Thus, if N is a d -dimensional Λ -module such that $M \leq_{deg} N$ as Λ -modules, then N is also a $\Lambda/\text{ann}(M)$ -module. Moreover, $M \leq_{deg} N$ as Λ -modules if and only if $M \leq_{deg} N$ as $\Lambda/\text{ann}(M)$ -modules. Consequently, given two d -dimensional Λ -modules M and N , where Λ is a finitely generated algebra, M degenerates to N if and only if there is an exact sequence of the form

$$0 \longrightarrow U \longrightarrow U \oplus M \longrightarrow N \longrightarrow 0$$

where $U \in \text{mod } \Lambda$. In fact, we can choose $U \in \text{mod } \Lambda/\text{ann}(M)$.

Clearly, each module degenerates to the semisimple module with the same composition factors. This can be seen using the exact sequence of the form

$$0 \longrightarrow \mathfrak{r}M \oplus \dots \oplus \mathfrak{r}^t M \longrightarrow M \oplus \mathfrak{r}M \oplus \dots \oplus \mathfrak{r}^t M \longrightarrow [M] \longrightarrow 0$$

where $\mathfrak{r}X$ is the radical of the module X , $\mathfrak{r}^{t+1}M = (0)$ and $[M]$ is the semisimple module with the same composition factors as M . We will denote by $\text{soc } X$ the socle of a module X .

From the equivalence of degeneration order with the existence of Riedtmann-sequences above, it is easy to see that if $M \leq_{deg} N$ then

$$\dim \text{Hom}_{\Lambda}(X, M) \leq \dim \text{Hom}_{\Lambda}(X, N)$$

for each Λ -module X . We will use the notation (X, Y) for $\text{Hom}_{\Lambda}(X, Y)$ and $[X, Y]$ for $\dim \text{Hom}_{\Lambda}(X, Y)$. More generally, we will use the notation $[F(X)]$ to denote the dimension of a finitely presented functor F applied to X , and even $[F(\cdot)]$ (or $[F]$) to indicate the element of the functor F in the Grothendieck group of the finitely presented functors on $\text{mod } \Lambda$.

If $[X, M] \leq [X, N]$ for each Λ -module X , we will denote this by $M \leq_{hom} N$. Auslander (see [1 T4.2]) has proved that $M \leq_{hom} N$ and $N \leq_{hom} M$ implies that M and N are isomorphic, so \leq_{hom} is a partial order on the set of isomorphism classes of d -dimensional modules.

These two partial orders are not equivalent. Degeneration-order implies hom-order as shown above, but Carlson gave an example (see [5]) of two modules M and N such that $M \leq_{hom} N$, while $M \not\leq_{deg} N$.

Riedtmann introduced in [5] still another partial order on the set of isomorphism classes of d -dimensional modules called the virtual degeneration order. We say that M virtually degenerates to N if there is some finite dimensional module W such that

$$M \oplus W \leq_{deg} N \oplus W,$$

and we denote this by $M \leq_{virt} N$. Virtual degeneration clearly implies hom-order, and for representation-finite algebras these two orders coincide [5]. It is an open question whether the hom-order and the virtual degeneration-order coincide in general.

Carlson's example is the first example known of a virtual degeneration between two modules where there is no degeneration, so that the virtual degeneration-order is strictly finer than the degeneration-order.

2. DEGENERATIONS AND FINITELY PRESENTED FUNCTORS

Given two Λ -modules X and Y , let $\text{rad}(Y, X)$ denote the subgroup of $\text{Hom}_\Lambda(Y, X)$ given by

$$\{f : Y \longrightarrow X \mid pfi \in \text{rad End}_\Lambda(Z), \forall i \in \text{Hom}_\Lambda(Z, Y), \forall p \in \text{Hom}_\Lambda(X, Z)\},$$

and denote by F_X the contravariant functor $\text{Hom}_\Lambda(\ , X)/\text{rad}(\ , X)$.

In other words, if X is an indecomposable finite dimensional module, then F_X is the simple contravariant functor given by $F_X(Y) \neq 0$ for an indecomposable module Y if and only if $Y \simeq X$. For $X = \bigoplus_{i=1}^t X_i$, where X_i is an indecomposable module for each i , F_X is the semisimple contravariant functor $\bigoplus_{i=1}^t F_{X_i}$.

For a given functor F , we will denote by $\text{soc } F$ its socle, and we will denote by $\text{r } F$ its radical.

Given two d -dimensional modules M and N such that $M \leq_{deg} N$ and consider a Riedtmann-sequence

$$\eta : \quad 0 \longrightarrow U \xrightarrow{f} U \oplus M \xrightarrow{g} N \longrightarrow 0,$$

giving this degeneration. The sequence η induces an exact sequence of contravariant functors

$$0 \longrightarrow (\ , U) \xrightarrow{(\ , f)} (\ , U \oplus M) \xrightarrow{(\ , g)} (\ , N) \xrightarrow{p} \delta \longrightarrow 0$$

where $\delta = (\ , N)/\text{Im}(\ , g)$ and p is the natural epimorphism. It is clear that $[\delta(X)] = [X, N] - [X, M]$ for each Λ -module X . Note that from the projective resolution of δ above, we get that the top of δ , $\delta/\text{r } \delta$, is isomorphic to $F_{N'}$ where N' is some summand of N .

Therefore if $M \leq_{deg} N$, there is a finitely presented functor δ such that $[\delta] = [\ , N] - [\ , M]$ and the top of δ corresponds to some summand of N .

Assume now that M and N are d -dimensional modules such that there exists a finitely presented contravariant functor δ with $[\delta] = [\ , N] - [\ , M]$. Note that this implies that $M \leq_{hom} N$. Given that M^* and N^* is another pair of d -dimensional modules such that $[\ , N] - [\ , M] = [\ , N^*] - [\ , M^*]$, we get that $N \oplus M^* \simeq M \oplus N^*$. If M and N have no common summands, it follows that $M^* = M \oplus T$ and $N^* = N \oplus T$ for some T . Also, if the greatest common summand of M and N is $W \neq (0)$, then $M = M' \oplus W$ and $N = N' \oplus W$ where M' and N' have no common summands,

and $M^* = M' \oplus T$ and $N^* = N' \oplus T$ for some T . Hence, the dimension vector $[\cdot, N] - [\cdot, M]$ is determined up to common direct summands by the pair M and N .

Let us consider a minimal projective resolution of δ . If the projective dimension of δ is less than two, i.e. if there is an exact sequence of contravariant functors

$$0 \longrightarrow (\cdot, X) \longrightarrow (\cdot, Y) \longrightarrow \delta \longrightarrow 0,$$

then

$$[\delta] = [\cdot, N] - [\cdot, M] = [\cdot, Y] - [\cdot, X],$$

and so $N \oplus X \simeq M \oplus Y$. It follows that X and Y have the same length, and since the projective resolution of δ above induces an inclusion from X to Y , we get that $X \simeq Y$ and then consequently that $M \simeq N$ and $\delta = 0$. Therefore if $\delta \neq 0$ then the projective dimension of δ is 2 and its projective resolution is of the form

$$0 \longrightarrow (\cdot, X) \longrightarrow (\cdot, Y) \longrightarrow (\cdot, Z) \longrightarrow \delta \longrightarrow 0$$

for some Λ -modules X, Y and Z .

Let $M = M' \oplus W$ and $N = N' \oplus W$, where M' and N' have no common nonzero summands. Then, since $[\delta] = [\cdot, N] - [\cdot, M]$, it follows that $X = N_1 \oplus S$, $Y = S \oplus M' \oplus T$ and $Z = N_2 \oplus T$ for some S and T where $N_1 \oplus N_2 = N'$, and we get an exact sequence of modules of the form

$$0 \longrightarrow N_1 \oplus S \longrightarrow S \oplus M' \oplus T \longrightarrow N_2 \oplus T \longrightarrow 0$$

By adding N_1 to the second and third term in this sequence with the identity map between them we get a Riedtmann-sequence for the degeneration $M' \oplus T \leq_{deg} N' \oplus T$, and by adding also W we get a Riedtmann-sequence for the degeneration $M \oplus T \leq_{deg} N \oplus T$. This gives the following proposition.

Proposition 2.1. *Given two d -dimensional modules M and N .*

(a) *M virtually degenerates to N if and only if there is some finitely presented functor $\delta : \text{mod } \Lambda \longrightarrow \text{mod } k$ such that $[\delta] = [\cdot, N] - [\cdot, M]$.*

(b) *M degenerates to N if and only if there is some finitely presented functor $\delta : \text{mod } \Lambda \longrightarrow \text{mod } k$ such that*

- (1) $[\delta] = [\cdot, N] - [\cdot, M]$ and
- (2) $\delta/\mathfrak{r}\delta \simeq F_{N'}$ where N' is some direct summand of N .

As pointed out by Riedtmann in [5], it is interesting to note that the question about equivalence between virtual degeneration and hom-order can be reformulated in the following way: given two d -dimensional modules M and N such that $M \leq_{hom} N$, is there always some finitely presented functor δ with $[\delta] = [\cdot, N] - [\cdot, M]$?

Recall (see [1, p. 6-7]) that a map $h : X \longrightarrow C$ in $\text{mod } \Lambda$ is right minimal if each map $w : X \longrightarrow X$ such that $h = hw$, has to be an isomorphism. For a given map $h : X \longrightarrow C$, there is a decomposition $X = X' \oplus X''$ such that $h|_{X'} : X' \longrightarrow C$ is right minimal and $h|_{X''} = 0$. The morphism $h|_{X'}$ is called a right minimal version of h .

Auslander has proved (see [1, Ch. XI]) the following proposition.

Proposition 2.2. *Given a finite dimensional algebra Λ , let*

$$0 \longrightarrow A \longrightarrow B \xrightarrow{h} C \longrightarrow 0$$

be a non-split exact sequence in $\text{mod } \Lambda$ and δ the contravariant functor induced by it. If h is right minimal then the socle of δ is isomorphic to the semisimple functor $F_{\text{TrD}(A)}$.

Let M and N be two d -dimensional Λ -modules such that $M \leq_{\text{deg}} N$ and let

$$\eta: 0 \longrightarrow U \xrightarrow{f} U \oplus M \xrightarrow{g} N \longrightarrow 0$$

be a Riedtmann-sequence giving this degeneration. If the map g is not right minimal, let $g^* : V' \longrightarrow N$ be a right minimal version of g and consider the exact sequence

$$0 \longrightarrow U' \xrightarrow{f^*} V' \xrightarrow{g^*} N \longrightarrow 0$$

where U' is the kernel of g^* and f^* is the inclusion of U' in V' . Note that U' is a summand of U . We have

$$[\ , M] = [\text{Im}(\ , g)] = [\text{Im}(\ , g^*)] = [\ , V'] - [\ , U'],$$

so that $[\ , V'] = [\ , U' \oplus M]$ and it follows that $V' \xrightarrow{\cong} U' \oplus M$. We then get the exact sequence of Riedtmann-form

$$\eta': 0 \longrightarrow U' \xrightarrow{f'} U' \oplus M \xrightarrow{g'} N \longrightarrow 0$$

where $f' = \phi f^*$ and $g' = g^* \phi^{-1}$, with g' right minimal. We call this sequence a right minimal version of the sequence we started with. Clearly, U' has no injective summands.

Corollary 2.3. *Let Λ be a finite dimensional algebra, and M and N two d -dimensional Λ -modules such that $M <_{\text{deg}} N$. If U_i is an indecomposable finitely generated module such that $[\text{TrD } U_i, M] = [\text{TrD } U_i, N]$, then U_i is not a summand of U for any exact sequence of the form*

$$0 \longrightarrow U \longrightarrow U \oplus M \longrightarrow N \longrightarrow 0$$

where the second map is right minimal.

In what follows we consider modules over Λ_q , the path algebra of the quiver

$$\Gamma: \alpha \begin{array}{c} \curvearrowright \\ \circlearrowleft \end{array} 1 \begin{array}{c} \curvearrowright \\ \circlearrowleft \end{array} \beta$$

with relations $\sigma_q = \{\alpha^2, \beta^2, \alpha\beta + q\beta\alpha\}$ over an algebraically closed field k , where $0 \neq q \in k$. Note that $\Lambda_q = k\langle x, y \rangle / \langle x^2, y^2, xy + qyx \rangle$.

For $q = -1$ and k with characteristic different from 2, the degenerations, the virtual degenerations and the hom-order for Λ_{-1} -modules are described using geometrical methods in [4].

In this paper, some types of virtual degenerations for Λ_q -modules will be described and situations in which one can cancel common direct summands will be given. We use the existence of exact sequences to give a degeneration, and we show that there is no sequence of Riedtmann-form to prove that there is no degeneration. There is no restriction on the characteristic of the field k in these considerations. In showing the non-existence of a Riedtmann-sequence for a given pair of modules, the information about U corresponding to the socle of a functor given in Proposition 2.2. will be used in two different ways. In one of them the result from the corollary above is used and in the other the socle-property described by Auslander is even more strongly used.

3. Λ_q -MODULES

The indecomposable Λ_q -modules (up to isomorphism) and almost split sequences between them are as follows (see [1, p. 362-363]):

- (1) The indecomposable projective module P_q .
- (2) The simple module $P_q/\mathfrak{r}P_q$, where $\mathfrak{r}P_q$ is the radical of P_q , and the pre-projective/preinjective modules $\mathrm{Tr}D^s(\mathfrak{r}P_q)$, $s \in \mathbb{Z}$ and $\mathrm{Tr}D^s(P_q/\mathfrak{r}P_q)$, $0 \neq s \in \mathbb{Z}$. We will denote these modules by ρ_s , $s \in \mathbb{Z}$, where ρ_0 is the simple module, $\rho_{2t-1} = \mathrm{Tr}D^t(\mathfrak{r}P_q)$ and $\rho_{2t} = \mathrm{Tr}D^t(\rho_0)$. The almost split exact sequences between modules of this form are:

$$0 \longrightarrow \rho_{-1} \longrightarrow P_q \oplus \rho_0^2 \longrightarrow \rho_1 \longrightarrow 0$$

and

$$0 \longrightarrow \rho_s \longrightarrow \rho_{s+1}^2 \longrightarrow \rho_{s+2} \longrightarrow 0$$

for $s \neq -1$.

- (3) Given $x = (x_1, x_2) \in \mathbb{P}^1$, we will identify x with $\frac{x_1}{x_2} \in k$ if $x_2 \neq 0$. If $x_2 = 0$, we identify x with ∞ . In particular, for a given $0 \neq c \in k$, $cx = \frac{cx_1}{x_2}$ if $x_2 \neq 0$, and $c\infty = \infty$.

For each $a \in \mathbb{P}^1$ and $n \in \mathbb{N}$, there is a $2n$ -dimensional indecomposable regular Λ_q -module (a, n) corresponding to a representation of (Γ, σ_q) given by

$$(a, n)(\alpha) = \begin{pmatrix} 0 & 0 \\ I_n & 0 \end{pmatrix}, \quad (a, n)(\beta) = \begin{pmatrix} 0 & 0 \\ J_n(a) & 0 \end{pmatrix}$$

for $a \neq \infty$. The module (∞, n) is given by a representation

$$(\infty, n)(\alpha) = \begin{pmatrix} 0 & 0 \\ J_n(0) & 0 \end{pmatrix}, \quad (\infty, n)(\beta) = \begin{pmatrix} 0 & 0 \\ I_n & 0 \end{pmatrix};$$

where I_n is the $n \times n$ -identity matrix and $J_n(x)$ is the Jordan block of size n and eigenvalue x .

Almost split sequences between the modules of this form are:

$$0 \longrightarrow (a, n) \longrightarrow (a, n+1) \oplus (a, n-1) \longrightarrow (a, n) \longrightarrow 0$$

where $(a, 0) = (0)$, the zero module, for each $a \in \mathbb{P}^1$.

We will call a module X regular if it is a direct sum of modules of this form.

The dimensions of the homomorphism spaces between these modules are given in the following table [4]:

$[X, Y]$	P_q	ρ_t	(b, m)
P_q	4	$2 t + 1$	$2m$
ρ_s	$2 s + 1$	$ s t + \frac{1}{2}(s + t + 1) + \frac{1}{2} s - t - 1 $	$(s + 1)m$
(a, n)	$2n$	$(t + 1)n$	$mn + \min\{m, n\}\delta_{a,b}$

where $\delta_{x,y}$ is the Kronecker δ defined by $\delta_{x,y} = 0$ if $x \neq y$ and $\delta_{x,y} = 1$ when $x = y$.

We denote $\dim_k \text{Ext}^1(X, Y)$ by ${}^1[X, Y]$ and give here the dimensions of the Ext^1 -spaces between the indecomposable modules:

${}^1[X, Y]$	P_q	ρ_t	(b, m)
P_q	0	0	0
ρ_s	0	$\frac{1}{2} s - t - 1 + \frac{1}{2} s - t - 2 + \frac{1}{2}$	m
(a, n)	0	n	$\min\{m, n\}\delta_{a,b} + \min\{m, n\}\delta_{a,qb}$

As we have seen, degenerations are closely connected to extensions of modules. We therefore analyze some extensions here in order to avoid too many digressions later.

Before we start, let us recall the following (see [1, Ch. X]): given an algebra Λ , we denote by $\underline{\text{mod}}\Lambda$ the category whose objects are those in $\text{mod}\Lambda$, and for a given pair of objects A and B , we have

$$\text{Hom}_{\underline{\text{mod}}\Lambda}(A, B) =: \underline{\text{Hom}}(A, B) = \text{Hom}_\Lambda(A, B)/\mathcal{P}(A, B)$$

where

$$\mathcal{P}(A, B) = \{h : A \longrightarrow B \in \text{mod}\Lambda \mid h \text{ factors through a projective } \Lambda\text{-module}\}.$$

Let Λ_{Kr} denote the Kronecker algebra, i.e. the path algebra of the quiver $1 \rightrightarrows 2$ over k . There is an equivalence $\theta : \underline{\text{mod}}(\Lambda_q/\text{soc}\Lambda_q) \longrightarrow \underline{\text{mod}}\Lambda_{Kr}$, and we say that $\Lambda_q/\text{soc}\Lambda_q$ and Λ_{Kr} are stably equivalent.

3.1. The sequences ζ . By [2, Section 5], using the stable equivalence above, we get that for each $s \geq 0$ and for each $y \in \mathbb{P}^1$, we have an exact sequence of the form

$$\zeta_y^+ : 0 \longrightarrow \rho_s \xrightarrow{j} \rho_{s+1} \longrightarrow (y, 1) \longrightarrow 0.$$

For $s < 0$, we shall prove that there is an exact sequence of the form

$$\zeta_y^- : 0 \longrightarrow \rho_s \xrightarrow{j} P_q \oplus \rho_{s+1} \longrightarrow (y, 1) \longrightarrow 0$$

for each $y \in \mathbb{P}^1$. By [2], there is for each $y \in \mathbb{P}^1$ and $s < 0$ an exact sequence of the form

$${}_y\zeta^- : 0 \longrightarrow (y, 1) \longrightarrow \rho_s \longrightarrow \rho_{s+1} \longrightarrow 0.$$

Given $s < 0$ and $y \in \mathbb{P}^1$, consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & (q^{-1}y, 1) & \xrightarrow{i} & P_q & \longrightarrow & (y, 1) \longrightarrow 0 \\ & & \parallel & & \uparrow h & & \uparrow \\ 0 & \longrightarrow & (q^{-1}y, 1) & \longrightarrow & \rho_s & \longrightarrow & \rho_{s+1} \longrightarrow 0 \end{array}$$

where the second row is the sequence ${}_{q^{-1}y}\zeta^-$, the map i is an injective envelope of $(q^{-1}y, 1)$ and h exists since P_q is injective. The second commutative square of this diagram gives an exact sequence

$$\zeta_y^- : 0 \longrightarrow \rho_s \longrightarrow P_q \oplus \rho_{s+1} \longrightarrow (y, 1) \longrightarrow 0$$

of the wanted form. We shall denote by

$$\zeta_y : 0 \longrightarrow \rho_s \xrightarrow{j} W \oplus \rho_{s+1} \longrightarrow (y, 1) \longrightarrow 0$$

both ζ_y^- and ζ_y^+ where $W = P_q$ when $s < 0$ and $W = 0$ otherwise.

3.2. The sequences κ_n and τ_n , for $n \geq 1$. Again using Bongartz' result in Section 5 in [2] and the stable equivalence above, we get that for each $a \in \mathbb{P}^1$ and each $n \geq 1$ there exists an exact sequence of the form

$$\kappa_n : 0 \longrightarrow \rho_{n-1} \longrightarrow (a, n) \longrightarrow \rho_0 \longrightarrow 0,$$

and also an exact sequence of the form

$$\tau_n : 0 \longrightarrow \rho_0 \longrightarrow \rho_n \longrightarrow (a, n) \longrightarrow 0.$$

3.3. The extensions of (b, m) by (a, n) . As seen in the Ext^1 -table above, $\text{Ext}^1((a, n), (b, m)) = 0$ if $a \neq b$ and $a \neq qb$. We have the following lemma concerning the middle term of a short exact sequence starting in (b, m) and ending in (a, n) .

Lemma 3.1. *Given a short exact sequence*

$$0 \longrightarrow (b, m) \xrightarrow{f} X \xrightarrow{g} (a, n) \longrightarrow 0,$$

we have that $X \simeq P_q^r \oplus (b, n_1) \oplus (a, n_2)$ for some $r, n_1, n_2 \in \mathbb{N}$ such that $n_1 + n_2 + 2c = m + n$ and $r \leq \min\{m, n\}$.

Proof. Of course, if $a \neq b$ and $a \neq qb$, the sequence above is split and the claim is clearly true. Therefore assume that $a = b$ or $a = qb$ in the rest of the proof.

Suppose that $X = (c, n') \oplus X'$ for some $c \notin \{b, qb\}$. From the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (b, m) & \xrightarrow{i} & P_q^m & \longrightarrow & (qb, m) & \longrightarrow & 0 \\ & & \parallel & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & (b, m) & \longrightarrow & (c, n') \oplus X' & \longrightarrow & (a, n) & \longrightarrow & 0 \end{array}$$

where the last row is the given sequence and i is an injective envelope of (b, m) , we get an exact sequence of the form

$$0 \longrightarrow (c, n') \oplus X' \longrightarrow P_q^m \oplus (a, n) \longrightarrow (qb, m) \longrightarrow 0.$$

Since $c \notin \{b, qb\}$, we know that $\text{Ext}^1((qb, n), (c, n')) = 0$. Thus (c, n') must split off in this sequence, and this is a contradiction since it is not a summand in the middle term. Therefore if X has an indecomposable summand of the form (x, n') , then $x = b$ or $x = qb$.

Assume now that there is some $s \in \mathbb{Z}$ such that $X = X' \oplus \rho_s$. Let $c \in \mathbb{P}^1$ be such that $c \notin \{b, qb, q^2b\}$ and consider the commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (b, m) & \longrightarrow & X' \oplus \rho_s & \longrightarrow & (a, n) & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (b, m) & \longrightarrow & X' \oplus W \oplus \rho_{s+1} & \longrightarrow & E & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & (c, 1) & \xlongequal{\quad} & (c, 1) & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

where the first row is the given sequence, the middle column is a sequence of the form ζ_c from Subsection 3.1, with X' added to the first two terms with the identity map between them, and E is the pushout. Now the last column is an extension of (a, n) by $(c, 1)$, and $a = b$ or $a = qb$ by assumption. Since $c \notin \{b, qb, q^2b\}$, this sequence must split and we get that $E \simeq (c, 1) \oplus (a, n)$. The middle row of the diagram is then an extension of (b, m) by $(c, 1) \oplus (a, n)$ and, again since $c \notin \{b, qb\}$, $(c, 1)$ must split off. This is a contradiction since it is not a summand of the middle term (it is not a summand of X as seen above). It follows that X has no summand of the form ρ_s , $s \in \mathbb{Z}$, i.e. each indecomposable summand of X is either projective or regular.

Let $c \notin \{a, b\}$ and assume that the module (c, i) is a summand of X , i.e. $X \simeq (c, i) \oplus (\oplus_j (a_j, m_j)) \oplus P_q^r$ for some $r \in \mathbb{N}$. Note that we always have $X \leq_{\text{hom}} (a, n) \oplus (b, m)$. But for $c \notin \{a, b\}$ we have

$$[(c, 1), (c, i) \oplus (\oplus_j (a_j, m_j)) \oplus P_q^r] > [(c, 1), (a, n) \oplus (b, m)],$$

and this gives a contradiction. Therefore $X \simeq P_q^r \oplus (\oplus_{i=1}^t (a, n_i)) \oplus (\oplus_{j=1}^s (b, m_j))$ for some $r, n_i, m_j \in \mathbb{N}$.

Suppose $a \neq b$. Then we have the inequality

$$2r + \sum_{i=1}^t n_i + t + \sum_{j=1}^s m_j = [(a, 1), X] \leq [(a, 1), (b, m) \oplus (a, n)] = m + n + 1,$$

and since $2r + \sum_{i=1}^t n_i + \sum_{j=1}^s m_j = m + n$, we get $t \leq 1$, i.e. X cannot have more than one summand of the form (a, n_i) . Similarly, there cannot be more than one summand of the form (b, m_j) .

For $a = b$, using that $[(b, 1), X] \leq [(b, 1), (b, m) \oplus (b, n)]$, we get that X cannot have more than two summands of the form (b, n_i) . Therefore X is isomorphic to $P_q^r \oplus (b, n_1) \oplus (a, n_2)$.

Finally, it is easy to see that r must be less than or equal to $\min\{m, n\}$: the top of (b, n) must be a summand of the top of X since g is an epimorphism. It follows that $n \leq r + n_1 + n_2$, giving $n + r \leq 2r + n_1 + n_2 = m + n$, i.e. $r \leq m$. Also, f is a monomorphism, so the socle of (b, m) must be a summand of the socle of X . Hence, $m \leq r + n_1 + n_2$ and $m + r \leq 2r + n_1 + n_2 = m + n$, i.e. $r \leq n$. \square

(1) $\text{Ext}^1((b, n), (b, m))$ when $b \neq qb$:

Given $m \leq n$, it is not difficult to see that for each $i = 1, \dots, m$, there is an exact sequence (arising from almost split sequences in the "b-tube" in the AR-quiver of Λ_q) of the form:

$$\xi_i : 0 \longrightarrow (b, m) \longrightarrow (b, m - i) \oplus (b, n + i) \longrightarrow (b, n) \longrightarrow 0.$$

Similarly, if $m \geq n$, for each $i = 1, \dots, n$, there is an exact sequence of the form

$$\xi_i : 0 \longrightarrow (b, m) \longrightarrow (b, m + i) \oplus (b, n - i) \longrightarrow (b, n) \longrightarrow 0.$$

Lemma 3.2. *If X is a middle term of a short exact sequence starting in (b, m) and ending in (b, n) , where $b \neq qb$, then X cannot have P_q as a summand.*

Proof. Assume that $X \simeq P_q \oplus X'$ and consider the following diagram where the second row is a sequence having X as a middle term and π is a projective cover of (b, n) :

$$\begin{array}{ccccccc} 0 & \longrightarrow & (q^{-1}b, n) & \longrightarrow & P_q^n & \xrightarrow{\pi} & (b, n) \longrightarrow 0 \\ & & \downarrow t & & \downarrow s & & \parallel \\ 0 & \longrightarrow & (b, m) & \xrightarrow{f} & P_q \oplus X' & \longrightarrow & (b, n) \longrightarrow 0 \end{array}$$

We get induced from this diagram an exact sequence of the form

$$0 \longrightarrow (q^{-1}b, n) \longrightarrow (b, m) \oplus P_q^n \xrightarrow{(f, s)} P_q \oplus X' \longrightarrow 0.$$

Note that given a map $\alpha : (x, i) \longrightarrow (y, j)$ where $x \neq y$, the image of α will be contained in the socle of (y, j) and the socle of (x, i) will be contained in the kernel of α .

Therefore, since $b \neq q^{-1}b$, the socle of $(q^{-1}b, n)$, which is isomorphic to ρ_0^n , will be contained in $\text{Ker } t = \text{Ker } s$. Now the socle of P_q^n is also isomorphic to ρ_0^n , and it follows that $\text{Im } s$ has Loewy length less than or equal to 2. Note that also $\text{Im } f$ has Loewy length less than or equal to 2. But then P_q cannot be a summand in $\text{Im}(f, s)$ giving the desired contradiction. \square

It follows from Lemma 3.1 that, when $b \neq qb$, the middle term of an exact sequence starting in (b, m) and ending in (b, n) must be of the form $(b, n_1) \oplus (b, n_2)$, where $n_1 + n_2 = m + n$.

Proposition 3.3. *Given an exact sequence of the form*

$$0 \longrightarrow (b, m) \longrightarrow (b, n_1) \oplus (b, n_2) \longrightarrow (b, n) \longrightarrow 0$$

with $n_1 \leq n_2$, we have $n_1 = \min\{m, n\} - i$ and $n_2 = \max\{m, n\} + i$ for some $i = 0, \dots, \min\{m, n\}$. In other words, if the sequence is not split, its middle term is isomorphic to the middle term of ξ_i for some i .

Proof. The claim clearly holds if the sequence is split ($i = 0$), so let us assume that it is not split. Assume that $m \leq n$ and that $n_1 = m + j$ for some $j > 0$. Since $n + m = n_1 + n_2$, it follows that $n_2 = n - j$. Let δ be the contravariant functor induced by the sequence above. Then (b, m) corresponds to the socle of δ , and consequently

$$[\delta(b, m)] = [(b, m), (b, m) \oplus (b, n)] - [(b, m), (b, m + j) \oplus (b, n - j)] > 0.$$

We get that $m(m + j) + m + m(n - j) + \min\{m, n - j\} < m^2 + m + mn + m$, i.e. $\min\{m, n - j\} < m$. Thus, $n_2 = n - j$ must be less than m . But this is a contradiction since $m + j = n_1 \leq n_2$ by assumption. Hence, the middle term is of the desired form.

The proof for $m \geq n$ is dual. \square

(2) Let us now consider $\text{Ext}^1((qb, n), (b, m))$ for $b \neq qb$.

Assume that $m \leq n$. We have the following commutative diagram for $j = 1, \dots, m - 1$:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & (b, n - m + j) & \xlongequal{\quad} & (b, n - m + j) & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & (b, n) & \longrightarrow & P_q^n & \longrightarrow & (qb, n) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & (b, m - j) & \longrightarrow & D & \longrightarrow & (qb, n) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

where the middle row is given by an injective envelope of (b, n) , the first column is a sequence of type ξ above and D is the pushout. By Schanuel's lemma we get that $D \simeq P_q^{m-j} \oplus (qb, n - m + j)$.

Consider now the following commutative diagram where the first row is the last row from the diagram above, the first column is again of the form ξ , and E is the

pushout:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (b, m-j) & \xrightarrow{h} & P_q^{m-j} \oplus (qb, n-m+j) & \xrightarrow{p} & (qb, n) \longrightarrow 0 \\
& & \downarrow i & & \downarrow & & \parallel \\
0 & \longrightarrow & (b, m) & \longrightarrow & E & \longrightarrow & (qb, n) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & (b, j) & \xlongequal{\quad\quad\quad} & (b, j) & & \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array}$$

Let G be the covariant functor induced by the first column of this diagram, i.e. given by the exact sequence

$$0 \longrightarrow ((b, j), \) \longrightarrow ((b, m), \) \xrightarrow{(i, \)} ((b, m-j), \) \longrightarrow G \longrightarrow 0.$$

It is easy to check that for X an indecomposable module, $[G(X)] \neq 0$ if and only if $X \simeq (b, r)$ where $r < m$. Therefore if X is not of this type, all maps $(b, m-j) \longrightarrow X$ will factor through i . Since $b \neq qb$ by assumption, the map h factors through the map i and then the middle column splits. The middle row is then an exact sequence of the form

$$0 \longrightarrow (b, m) \longrightarrow P_q^{m-j} \oplus (qb, n-m+j) \oplus (b, j) \longrightarrow (qb, n) \longrightarrow 0,$$

which for $r = m-j$ is of the form

$$0 \longrightarrow (b, m) \longrightarrow P_q^r \oplus (b, m-r) \oplus (qb, n-r) \longrightarrow (qb, n) \longrightarrow 0.$$

We will denote these sequences by ν_r , $r = 1, \dots, m-1$.

For $m = n$, let ν_m be the sequence

$$0 \longrightarrow (b, m) \longrightarrow P_q^m \longrightarrow (qb, m) \longrightarrow 0$$

where the first map is an injective envelope of (b, m) . For $m < n$, let ν_m be the last row of the previous diagram for $j = 0$, i.e.

$$0 \longrightarrow (b, m) \longrightarrow P_q^m \oplus (qb, n-m) \longrightarrow (qb, n) \longrightarrow 0.$$

If $m > n$, one can similarly as above show that there are sequences of the form

$$\nu_r : 0 \longrightarrow (b, m) \longrightarrow P_q^r \oplus (b, m-r) \oplus (qb, n-r) \longrightarrow (qb, n) \longrightarrow 0$$

for $r = 1, \dots, n$.

Therefore, for $b \neq qb$, there are short exact sequences of the form ν_r , $r = 1, \dots, \min\{m, n\}$, starting in (b, m) and ending in (qb, n) . We show in the following proposition that the middle terms of the ν_r 's are the only possible middle terms (up to isomorphism) when $b \neq qb$.

Proposition 3.4. *Let*

$$0 \longrightarrow (b, m) \xrightarrow{f} X \xrightarrow{g} (qb, n) \longrightarrow 0$$

be a non-split exact sequence where $b \neq qb$. Then X is isomorphic to the module $P_q^r \oplus (qb, n-r) \oplus (b, m-r)$ for some $r = 1, \dots, \min\{m, n\}$.

Proof. By Lemma 3.1, $X \simeq P_q^l \oplus (b, n_1) \oplus (qb, n_2)$ where $2l + n_1 + n_2 = m + n$ and $l \leq \min\{m, n\}$. Since $b \neq qb$, the socle of (b, m) , which is isomorphic to ρ_0^m , is contained in the kernel of the coordinate $f_3 : (b, m) \rightarrow (qb, n_2)$ of f . Since f is a monomorphism, we get that ρ_0^m must be a submodule of $P_q^l \oplus (b, n_1)$. It follows that $l + n_1 \geq m$.

Similarly, the image of the coordinate $g_2 : (b, n_1) \rightarrow (qb, n)$ of g is contained in the socle of (qb, n) . Since g is an epimorphism it follows that the top of (qb, n) , which is isomorphic to ρ_0^n , must be a summand of the top of $P_q^l \oplus (qn, n_2)$, which is isomorphic to $\rho_0^{l+n_2}$. It follows that $l + n_2 \geq n$.

Now we have that $2l + n_1 + n_2 = m + n$, $l + n_1 \geq m$ and $l + n_2 \geq n$. Hence $m = l + n_1$, $n = l + n_2$ and X is isomorphic to the middle term of a sequence of the form ν_l , $l = 1, \dots, \min\{m, n\}$. \square

(3) Let us now consider $\text{Ext}^1((b, n), (b, m))$ when $b = qb$.

It is clear that there are sequences of the form ξ_i , $i = 1, \dots, \min\{m, n\}$ and a sequence of the form $\nu_{\min\{m, n\}}$ also in this case, but it is slightly more complicated to see that we also have sequences of the form ν_r , $r = 1, \dots, \min\{m, n\} - 1$: Namely, in the last diagram, we use that $b \neq qb$ to show that h factors through i . Here we have that $b = qb$, but h still factors through i as we will now see. Let us assume that $m \leq n$; the proof is dual for $m > n$. Consider this last diagram once more, now for $b = qb$:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & (b, m-j) & \xrightarrow{h} & P_q^{m-j} \oplus (b, n-m+j) & \xrightarrow{p} & (b, n) \longrightarrow 0 \\
 & & \downarrow i & & \downarrow & & \parallel \\
 0 & \longrightarrow & (b, m) & \longrightarrow & E & \longrightarrow & (b, n) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & (b, j) & \xlongequal{\quad\quad\quad} & (b, j) & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

Let $h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$. Clearly, h_1 factors through i . Note that the top of the middle term of the first row is isomorphic to the top of the last term. Since p is an epimorphism we get that $\text{Ker } p$ is contained in the radical of the middle term. In particular, $\text{Im } h_2$ is contained in the radical of $(b, n-m+j)$, which is isomorphic to ρ_0^{n-m+j} . But all maps $(b, m-j) \rightarrow \rho_0$ factor through i , since $G(\rho_0) = 0$ (G is, as in (2), the covariant functor induced by the first column). Consequently, the map h_2 also factors through i , and there are sequences of the form ν_r , $r = 1, \dots, \min\{m, n\}$ also in this case.

Therefore, for $b = qb$, there are sequences of the form

$$\{\xi_i | i = 1, \dots, \min\{n, m\}\} \text{ and } \{\nu_r | r = 1, \dots, \min\{n, m\}\}$$

in $\text{Ext}^1((b, n), (b, m))$, where ξ_i and ν_r are as in (1) and (2), respectively.

We have seen that the middle terms of the ξ_i 's are the only possible middle terms (up to isomorphism) in case (1), and also that the middle terms of the ν_r 's are the only possible middle terms (up to isomorphism) in case (2).

We will give here a full description of the middle terms of extensions of (b, m) by (b, n) when $b = qb$. As we will see, in this case there are not just middle terms of ξ_i 's and ν_r 's appearing.

Lemma 3.5. *Given $b = qb$, there is an extension of the form*

$$0 \longrightarrow (b, m) \longrightarrow P_q^r \oplus X \longrightarrow (b, n) \longrightarrow 0$$

for each $1 \leq r < \min\{n, m\}$ and each regular module X such that there is an exact sequence of the form

$$0 \longrightarrow (b, m - r) \longrightarrow X \longrightarrow (b, n - r) \longrightarrow 0.$$

Recall that, by Lemma 3.1 and Proposition 3.3, if $m \leq n$, then X is isomorphic to $(b, m - r - i) \oplus (b, n - r + i)$ for $i = 0, \dots, m - r$, and, if $m \geq n$, then X is isomorphic to $(b, m - r + i) \oplus (b, n - r - i)$ for some $i = 0, \dots, n - r$. In other words, the extension of $(b, m - r)$ by $(b, n - r)$ above is of the form ξ .

Proof. The claim is proved by diagram chasing. For some $1 \leq r < \min\{n, m\}$, consider the following diagram:

$$\begin{array}{ccccccc}
 & & 0 & & & & 0 \\
 & & \downarrow & & & & \downarrow \\
 0 & \longrightarrow & (b, r) & \xrightarrow{j} & P_q^r & \xrightarrow{\pi} & (b, r) \longrightarrow 0 \\
 & & \downarrow i_1 & \nearrow g & \searrow f & & \downarrow p_3 \\
 & & (b, m) & & & & (b, n) \\
 & & \downarrow p_1 & & & & \downarrow i_3 \\
 0 & \longrightarrow & (b, m - r) & \xrightarrow{i_2} & X & \xrightarrow{p_2} & (b, n - r) \longrightarrow 0 \\
 & & \downarrow & & & & \downarrow \\
 & & 0 & & & & 0
 \end{array}$$

where j is an injective envelope of (b, r) , the columns and the last row are sequences of the form ξ . The maps g and f exist since P_q^r is an injective and a projective module, respectively.

It is easy to see that the map $\begin{pmatrix} g \\ i_2 p_1 \end{pmatrix} : (b, m) \longrightarrow P_q^r \oplus X$ is a monomorphism and the map $(f, i_3 p_2) : P_q^r \oplus X \longrightarrow (b, n)$ is an epimorphism by construction, but $fg \neq i_3 p_2 i_2 p_1 = 0$. We shall find two maps, $h_1 : (b, m) \longrightarrow X$ and $h_2 : X \longrightarrow (b, n)$ such that $\begin{pmatrix} g \\ i_2 p_1 + h_1 \end{pmatrix} : (b, m) \longrightarrow P_q^r \oplus X$ is a monomorphism, $(f, i_3 p_2 + h_2) : P_q^r \oplus X \longrightarrow (b, n)$ is an epimorphism and $fg = (i_3 p_2 + h_2)(i_2 p_1 + h_1)$. It will then follow that

$$0 \longrightarrow (b, m) \xrightarrow{\begin{pmatrix} -g \\ i_2 p_1 + h_1 \end{pmatrix}} P_q^r \oplus X \xrightarrow{(f, i_3 p_2 + h_2)} (b, n) \longrightarrow 0$$

is an exact sequence, and our claim will be proved.

Since $p_3 f j = \pi j = 0$, it follows that $\text{Im } f j \subseteq \text{Ker } p_3 = \text{Im } i_3$ and thus there is a unique map $w : (b, r) \rightarrow (b, n - r)$ such that $i_3 w = f j$. Note that the radical of (b, r) is contained in the kernel of $f j$, and hence is contained in the kernel of w . Let $\bar{w} : (b, r)/\mathfrak{r}(b, r) \rightarrow (b, n - r)$ be such that $w = \bar{w} \pi_r$, where $\pi_r : (b, r) \rightarrow (b, r)/\mathfrak{r}(b, r)$ is a natural epimorphism. Consider the commutative diagram

$$\begin{array}{ccc} (b, r)/\mathfrak{r}(b, r) & \xleftarrow{\pi_r} & (b, r) \\ \downarrow i'_1 & & \downarrow i_1 \\ (b, m)/\mathfrak{r}(b, m) & \xleftarrow{\pi_m} & (b, m) \end{array}$$

where π_m is a natural epimorphism. Clearly, i'_1 is a split monomorphism. Let $j'_1 : (b, m)/\mathfrak{r}(b, m) \rightarrow (b, r)/\mathfrak{r}(b, r)$ be such that $j'_1 i'_1$ is the identity map on $(b, r)/\mathfrak{r}(b, r)$ and let $w_2 := \bar{w} j'_1 \pi_m : (b, m) \rightarrow (b, n - r)$.

Note that the image of $f j$ is contained in the socle of (b, n) . It is therefore easy to see that the image of w_2 is contained in the socle of $(b, n - r)$. Now consider the following commutative square:

$$\begin{array}{ccc} \text{soc } X & \xrightarrow{s_x} & X \\ \downarrow p'_2 & & \downarrow p_2 \\ \text{soc}(b, n - r) & \xrightarrow{s_{n-r}} & (b, n - r) \end{array}$$

where s_{n-r} and s_x are natural monomorphisms. Since X is a regular module, p'_2 is a split epimorphism. Let $q'_2 : \text{soc}(b, n - r) \rightarrow \text{soc } X$ be such that $p'_2 q'_2 = \text{id}$. Since $\text{Im } w_2 \subseteq \text{soc}(b, n - r)$, we can define $h_1 : (b, m) \rightarrow X$ by $h_1 = q'_2 w_2$. It follows that $p_2 h_1 = w_2$.

Also note that h_1 has the radical of (b, m) contained in its kernel. It follows that $\binom{g}{i_2 p_1 + h_1} : (b, m) \rightarrow P_q^r \oplus X$ is a monomorphism: $\binom{g}{i_2 p_1} : (b, m) \rightarrow P_q^r \oplus X$ is a monomorphism, so $\text{soc}(\text{Ker } g) \cap \text{soc}(\text{Ker}(i_2 p_1)) = (0)$. However,

$$\begin{aligned} \text{Ker}(i_2 p_1) \cap \text{soc}(a, m) &= \text{soc}(\text{Ker}(i_2 p_1)) = \text{soc}(\text{Ker}(i_2 p_1 + h_1)) \\ &= \text{Ker}(i_2 p_1 + h_1) \cap \text{soc}(a, m) \end{aligned}$$

and consequently $\text{soc}(\text{Ker } g) \cap \text{soc}(\text{Ker}(i_2 p_1 + h_1)) = (0)$ and $\binom{g}{i_2 p_1 + h_1}$ is a monomorphism.

Now

$$f g i_1 = f j = i_3 \bar{w} \pi_r = i_3 \bar{w} j'_1 \pi_m i_1 = i_3 w_2 i_1 = i_3 p_2 h_1 i_1.$$

Therefore $(f g - i_3 p_2 h_1) i_1 = 0$, and hence there is a unique map $v : (b, m - r) \rightarrow (b, n)$ such that $v p_1 = f g - i_3 p_2 h_1$. Note that the radical of $(b, m - r)$ is contained in the kernel of v and also that the image of v is contained in the socle of (b, n) . Let $\bar{v} : (b, m - r)/\mathfrak{r}(b, m - r) \rightarrow (b, n)$ be such that $v = \bar{v} \pi_{m-r}$, where $\pi_{m-r} : (b, m - r) \rightarrow (b, m - r)/\mathfrak{r}(b, m - r)$ is a natural epimorphism. Consider the commutative diagram

$$\begin{array}{ccc} (b, m - r) & \xrightarrow{i_2} & X \\ \downarrow \pi_{m-r} & & \downarrow \pi_x \\ (b, m - r)/\mathfrak{r}(b, m - r) & \xrightarrow{i'_2} & X/\mathfrak{r} X \end{array}$$

where π_{m-r} and π_x are natural epimorphisms. Since X is a regular module, i_2' is a split monomorphism. Let the map $j_2' : X/\tau X \rightarrow (b, m-r)/\tau(b, m-r)$ be such that $j_2' i_2' = id$, and let $h_2 := \bar{v} j_2' \pi_x : X \rightarrow (b, n)$. The image of h_2 is contained in the socle of (a, n) , and similarly as above (in showing that $\binom{g}{i_2 p_1 + h_1}$ is a monomorphism), the map $(f, i_3 p_2 + h_2) : P_q^r \oplus X \rightarrow (b, n)$ is an epimorphism.

Since

$$h_2 i_2 = \bar{v} j_2' \pi_x i_2 = \bar{v} \pi_{m-r} = v$$

we get equalities

$$h_2 i_2 p_1 = v p_1 = f g - i_3 p_2 h_1.$$

The image of h_1 is contained in the socle of X and the socle of X is contained in the kernel of h_2 , so $h_2 h_1 = 0$ and

$$f g = (i_3 p_2 + h_2)(i_2 p_1 + h_1)$$

as desired. \square

We will now show that the middle terms which we have seen by now are the only ones possible (up to isomorphism) for extensions of (b, m) by (b, n) when $b = qb$.

Proposition 3.6. *Let $b = qb$ and let*

$$0 \rightarrow (b, m) \xrightarrow{f} X \xrightarrow{g} (b, n) \rightarrow 0$$

be an exact sequence.

If $m \leq n$, then X is isomorphic to $P_q^r \oplus (b, m-r-i) \oplus (b, n-r+i)$ for some $r = 0, \dots, m$ and $i = 0, \dots, m-r$. Similarly, if $m \geq n$, X is isomorphic to $P_q^r \oplus (b, m-r+i) \oplus (b, n-r-i)$ for some $r = 0, \dots, n$ and $i = 0, \dots, n-r$.

Proof. The claim clearly holds if the sequence is split, so assume that it is not split. We give the proof for $m \leq n$. The proof for $m \geq n$ is dual.

By Lemma 3.1, X is isomorphic to $P_q^r \oplus (b, n_1) \oplus (b, n_2)$ for some $r, n_1, n_2 \in \mathbb{N}$ such that $2r + n_1 + n_2 = m + n$ and $r \leq m = \min\{m, n\}$. We shall first show that if g is an epimorphism, then we must have $r + \max\{n_1, n_2\} \geq n$. Assume that $r + \max\{n_1, n_2\} < n$. Let $g = (g_3, g_1, g_2) : P_q^r \oplus (b, n_1) \oplus (b, n_2) \rightarrow (b, n)$ and let $n_1 \geq n_2$.

Let Λ_{K_r} denote, as usual, the Kronecker algebra and let $\theta : \underline{\text{mod}}(\Lambda_q/\text{soc } \Lambda_q) \rightarrow \underline{\text{mod}}\Lambda_{K_r}$ be the stable equivalence between Λ_{K_r} and $\Lambda_q/\text{soc } \Lambda_q$. Then (see [1]) $\theta(a, m) = R_{(a, m)}$ is the indecomposable regular Λ_{K_r} -module given as the representation

$$\begin{array}{ccc} & I & \\ k^m & \xrightarrow{\quad} & k^m \\ & J_m(a) & \end{array}$$

where I is the $m \times m$ -identity matrix, and $J_m(a)$ is the Jordan block of size m with eigenvalue a .

If $n_1 = n_2 = 0$, then $r = n$. Since we assume that $r + \max\{n_1, n_2\} < n$ and $n_1 \geq n_2$, we get $1 \leq n_1 < n$ and there is then an exact sequence of the form

$$0 \rightarrow (b, n_1) \xrightarrow{i} (b, n) \rightarrow (b, n - n_1) \rightarrow 0.$$

The map $\theta(g_2)$ factors through the map $\theta(i)$ (see [1, VIII.7]), i.e. there is a map $h_2' : R_{(b, n_2)} \rightarrow R_{(b, n_1)}$ such that $\theta(g_2) = \theta(i)h_2'$. But then the map $h_2 := \theta^{-1}(h_2') : (b, n_2) \rightarrow (b, n_1)$ is such that $g_2 - i h_2 = w_2 : (b, n_2) \rightarrow (b, n)$ is zero in $\underline{\text{mod}}(\Lambda_q/\text{soc } \Lambda_q)$, i.e. w_2 factors through some projective $\Lambda_q/\text{soc } \Lambda_q$ -module

$(\Lambda_q/\text{soc } \Lambda_q)^l$, $l \geq 1$. It follows that the image of w_2 is contained in the radical of $(\Lambda_q/\text{soc } \Lambda_q)^l$ and the radical of $(\Lambda_q/\text{soc } \Lambda_q)^l$ is a semisimple module. Thus, the image of w_2 is contained in the socle of (b, n) .

Similarly, using the equivalence θ , there exists a map $h_1 : (b, n_1) \rightarrow (b, n_1)$ such that $g_1 - ih_1 = w_1 : (b, n_1) \rightarrow (b, n)$ is zero in $\underline{\text{mod}}(\Lambda_q/\text{soc } \Lambda_q)$, and, similarly as above, the image of w_1 is then contained in $\text{soc}(b, n)$. Therefore we have that

$$(g_1 \ g_2) = (i \ j) \begin{pmatrix} h_1 & h_2 \\ w_1 & w_2 \end{pmatrix}$$

where j is the natural inclusion $\text{soc}(b, n) \rightarrow (b, n)$. It follows that $\text{Im}(g_1 \ g_2) \subseteq \text{Im}(i \ j)$. Also $(a, n)/\text{Im}(i \ j) \simeq \rho_0^{n-n_1}$ and hence $\rho_0^{n-n_1}$ is a summand of the top of $(a, n)/\text{Im}(g_1 \ g_2)$. Since $r < n - n_1$ by assumption, it follows that g cannot be an epimorphism. Thus, since g is an epimorphism by assumption, we must have that $r + \max\{m, n\} \geq n$.

Assume now that $n_1 = m - r + j$ for some $j > 0$. It follows that $n_2 = n - r - j$. If $n_2 \geq n_1$, we get that $r + \max\{n_1, n_2\} = r + n - r - j = n - j \geq n$, giving a contradiction since $j > 0$ by assumption.

Therefore we must have $n_1 \geq n_2$, and then $r + \max\{n_1, n_2\} = r + m - r + j \geq n$, i.e. $m + j \geq n$. Let $k \geq 0$ be such that $n + k = m + j$. Note that $k \leq m - r$ ($n_2 < 0$ otherwise, a contradiction). It follows that $n_1 = m - r + j = n - r + k$ and then $n_2 = m - r - k$, where $0 \leq k \leq m - r$, and the claim is proved. \square

4. SOME VIRTUAL DEGENERATIONS

Given two d -dimensional Λ_q -modules M and N it is easy, using the dimension table for homomorphisms, to determine whether $M \leq_{\text{hom}} N$ or not. To decide whether $M \leq_{\text{deg}} N$ is not that easy, especially if P_q is a summand of M , as we will now see.

Assume that P_q is neither a summand of M nor a summand of N . Then M and N are $\Lambda_q/\text{soc } \Lambda_q$ -modules, and $\Lambda_q/\text{soc } \Lambda_q$ is stably equivalent to Λ_{Kr} , the Kronecker algebra. By [2, Section 5], degeneration- and hom-order coincide for Λ_{Kr} -modules. And, by [5, Proposition 4.7], these two orders coincide then also for $\Lambda_q/\text{soc } \Lambda_q$ -modules, i.e. $M \leq_{\text{deg}} N$ if and only if $M \leq_{\text{hom}} N$ when P_q is not a summand of $M \oplus N$. In particular, we can always cancel common direct summands in degenerations between modules of this type.

Using the characterization of a degeneration by existence of a Riedtmann-sequence, it is easy to see that if P_q is a summand of N and $M \leq_{\text{deg}} N$, then P_q must be a summand of M too, and it can be cancelled from this degeneration since it is a projective module. Therefore the only situation left to consider is the following: given two d -dimensional Λ_q -modules M and N such that P_q is a summand of M and $M \leq_{\text{hom}} N$, does M virtually degenerate to N and does M degenerate to N ?

For $q = -1$, Carlson showed that P_q virtually degenerates to $(a, 1) \oplus (b, 1)$ for all $a, b \in \mathbb{P}^1$, but he argued that P_q cannot degenerate to each $(a, 1) \oplus (b, 1)$. He used the following argument: The dimension of the orbit of P_q , and its closure too, is 12 and the dimension of the orbit of $(a, 1) \oplus (b, 1)$ when $a \neq b$ is 10. Now the union of orbits of $(a, 1) \oplus (b, 1)$ for all a and b from \mathbb{P}^1 is a constructible set of the dimension 12, and therefore it cannot be contained in the closure of the orbit of P_q .

In this section we shall give some virtual degenerations of this type, having P_q as a direct summand of M . This is easy, one "just" has to find some proper Riedtmann-sequences and use transitivity. The more difficult part is to decide whether or not it is possible to cancel common summands. This will be done in the next section.

Proposition 4.1. *For all $n \geq 2$ and for all $a \in \mathbb{P}^1$, there is a degeneration*

$$P_q \oplus (a, n - 2) \oplus \rho_0 \leq_{deg} (a, n) \oplus \rho_0,$$

and a degeneration

$$P_q \oplus (a, n) \oplus (a, n - 2) \oplus \rho_0 \leq_{deg} (a, n + 1) \oplus (a, n - 1) \oplus \rho_0.$$

Proof. Let $n = 2$. Consider the following commutative diagram where the first row is given by an injective envelope of $(a, 1)$, and the second row is a sequence of the form ζ_{qa}^+ from Subsection 3.1:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (a, 1) & \longrightarrow & P_q & \longrightarrow & (qa, 1) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & \rho_0 & \longrightarrow & \rho_1 & \longrightarrow & (qa, 1) & \longrightarrow & 0 \end{array}$$

We get induced an exact sequence of the form

$$0 \longrightarrow (a, 1) \longrightarrow P_q \oplus \rho_0 \longrightarrow \rho_1 \longrightarrow 0,$$

and it follows that

$$P_q \oplus \rho_0 \leq_{deg} (a, 1) \oplus \rho_1$$

for all a .

Assume now that $n \geq 3$ and consider the commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & (a, 1) & \xrightarrow{j} & P_q \oplus \rho_0 & \longrightarrow & \rho_1 & \longrightarrow & 0 \\ & & \downarrow h & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & (a, n - 1) & \longrightarrow & D & \longrightarrow & \rho_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & (a, n - 2) & \equiv & (a, n - 2) & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

where the first row is the sequence we got above, the first column is a sequence of the form ξ and D is the pushout.

Note that $[G(P_q \oplus \rho_0)] = 0$ where G is the covariant functor induced by the first column:

$$0 \longrightarrow ((a, n - 2),) \longrightarrow ((a, n - 1),) \longrightarrow ((a, 1),) \longrightarrow G \longrightarrow 0.$$

It follows that the map j factors through the map h . Consequently, the middle column of the diagram splits, and $D \simeq P_q \oplus \rho_0 \oplus (a, n-2)$. The middle row now gives that

$$P_q \oplus (a, n-2) \oplus \rho_0 \leq_{deg} (a, n-1) \oplus \rho_1$$

for all a and all $n \geq 3$.

For $n \geq 2$, we have $\rho_1 \oplus (a, n-1) \leq_{hom} (a, n) \oplus \rho_0$. Since degeneration- and hom-order coincide for this type of modules by the remark at the beginning of this section, we have a degeneration

$$\rho_1 \oplus (a, n-1) \leq_{deg} (a, n) \oplus \rho_0.$$

By the transitivity of the degeneration order, we then get that for all a and all $n \geq 2$

$$P_q \oplus (a, n-2) \oplus \rho_0 \leq_{deg} (a, n) \oplus \rho_0.$$

The second degeneration from the proposition follows directly from here: there is a degeneration $P_q \oplus (a, n-1) \oplus \rho_0 \leq_{deg} (a, n+1) \oplus \rho_0$ as we have seen above. Clearly, we then have a degeneration

$$P_q \oplus (a, n-1) \oplus (a, n-1) \oplus \rho_0 \leq_{deg} (a, n+1) \oplus (a, n-1) \oplus \rho_0.$$

Note that $(a, n) \oplus (a, n-2)$ degenerates to $(a, n-1) \oplus (a, n-1)$ (given by an almost split sequence starting in $(a, n-1)$). It follows that

$$P_q \oplus (a, n) \oplus (a, n-2) \oplus \rho_0 \leq_{deg} (a, n+1) \oplus (a, n-1) \oplus \rho_0$$

for all $a \in \mathbb{P}^1$, $n \geq 2$. \square

Proposition 4.2. *For all $a, b \in \mathbb{P}^1$ and all $n \geq 1$,*

$$(1) \quad P_q^n \oplus \rho_0 \leq_{deg} (a, 2n) \oplus \rho_0$$

and

$$(2) \quad P_q^n \oplus \rho_0 \leq_{deg} (a, n) \oplus (b, n) \oplus \rho_0.$$

Proof. (1) Since for $a \in \mathbb{P}^1$ and each $n \geq 1$ we have $\rho_n \oplus (a, n) \leq_{hom} (a, 2n) \oplus \rho_0$, we get, similarly as before, that

$$\rho_n \oplus (a, n) \leq_{deg} (a, 2n) \oplus \rho_0.$$

On the other hand, from the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & (a, n) & \longrightarrow & P_q^n & \longrightarrow & (qa, n) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \rho_0 & \longrightarrow & \rho_n & \longrightarrow & (qa, n) \longrightarrow 0 \end{array}$$

we get induced an exact sequence of the form

$$0 \longrightarrow (a, n) \longrightarrow P_q^n \oplus \rho_0 \longrightarrow \rho_n \longrightarrow 0,$$

and consequently

$$P_q^n \oplus \rho_0 \leq_{deg} (a, n) \oplus \rho_n.$$

By transitivity of the degeneration order, it follows that

$$P_q^n \oplus \rho_0 \leq_{deg} (a, 2n) \oplus \rho_0$$

for all $a \in \mathbb{P}^1$.

(2) As seen in Subsection 3.2, for each $a \in \mathbb{P}^1$ and each $n \geq 1$, there is an exact sequence of the form κ_n :

$$0 \longrightarrow \rho_{n-1} \longrightarrow (a, n) \longrightarrow \rho_0 \longrightarrow 0.$$

From the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \rho_{n-1} & \xrightarrow{i} & P_q^n & \longrightarrow & \rho_n \longrightarrow 0 \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \rho_{n-1} & \longrightarrow & (a, n) & \longrightarrow & \rho_0 \longrightarrow 0 \end{array}$$

where the last row is an exact sequence of the form κ_n and i is an injective envelope, we get induced an exact sequence of the form

$$0 \longrightarrow (a, n) \longrightarrow P_q^n \oplus \rho_0 \longrightarrow \rho_n \longrightarrow 0.$$

This gives that

$$P_q^n \oplus \rho_0 \leq_{deg} (a, n) \oplus \rho_n$$

for all a .

Also, for each $b \in \mathbb{P}^1$ and each $n \geq 1$ we have $\rho_n \leq_{hom} (b, n) \oplus \rho_0$, and consequently $\rho_n \leq_{deg} (b, n) \oplus \rho_0$ for all b . By transitivity of degeneration order, it follows that

$$P_q^n \oplus \rho_0 \leq_{deg} (a, n) \oplus (b, n) \oplus \rho_0$$

for all $a, b \in \mathbb{P}^1$. \square

5. CANCELLING COMMON SUMMAND?

We start by giving two lemmas which will be used in the proofs of the main results in this section, Theorems 5.3 and 5.4.

Lemma 5.1. *Given two d -dimensional Λ_q -modules M and N such that $M <_{deg} N$, let $0 \longrightarrow U \xrightarrow{f} U \oplus M \xrightarrow{g} N \longrightarrow 0$, where g is right minimal, be a Riedtmann-sequence giving this degeneration. If ρ_{s+1} is not a summand of N , $[\rho_{s+2}, N] - [\rho_{s+2}, M] = 1$ and $[\rho_{s+1}, N] - [\rho_{s+1}, M] = 1$, then ρ_s is not a summand of U . In particular, if N is a regular module and $[\rho_s, N] - [\rho_s, M] = 1$ for all $s \in \mathbb{Z}$, then U is a regular module.*

Proof. Let δ be the finitely presented contravariant functor induced by the sequence above, and let $s \in \mathbb{Z}$ be such that ρ_{s+1} is not a summand of N and $[\rho_{s+i}, N] - [\rho_{s+i}, M] = 1$ for $i = 1, 2$. Assume that ρ_s is a summand in U .

Since $[\delta(\rho_{s+2})] = 1$, there is a nonzero map $h : \rho_{s+2} \longrightarrow N$ which does not factor through the map g . It follows that $p(h) \neq 0$ in $\delta(\rho_{s+2})$, where p is the natural epimorphism

$$0 \longrightarrow (, U) \xrightarrow{(\cdot, f)} (, U \oplus M) \xrightarrow{(\cdot, g)} (, N) \xrightarrow{p} \delta \longrightarrow 0$$

By Yoneda's lemma we have

$$\delta(\rho_{s+2}) \xrightarrow{\phi} \text{Hom}((, \rho_{s+2}), \delta)$$

and so $\phi(p(h)) : (, \rho_{s+2}) \longrightarrow \delta$ is nonzero and $0 \neq \text{Im } \phi(p(h)) \subset \delta$ is a subfunctor of δ . Note that since $[\delta(\rho_{s+2})] = 1$, any non-zero element from $\delta(\rho_{s+2})$ will generate the same subfunctor, $\text{Im } \phi(p(h))$, in δ . We call this subfunctor $\langle \rho_{s+2} \rangle$. Note also

that $\langle \rho_{s+2} \rangle$ has a simple top $F_{\rho_{s+2}}$ since, by definition of $\langle \rho_{s+2} \rangle$, we have an exact sequence of functors

$$(\ , \rho_{s+2}) \longrightarrow \langle \rho_{s+2} \rangle \longrightarrow 0.$$

Consider the almost split sequence having ρ_{s+2} as a summand in the middle term:

$$0 \longrightarrow \rho_{s+1} \xrightarrow{j} \rho_{s+2}^2 \oplus Z \xrightarrow{\pi} \rho_{s+3} \longrightarrow 0.$$

Note that $Z = (0)$ except for $s+1 = -1$ as we have seen before. If $s+1 = -1$ then $Z = P_q$, and in both cases $\delta(Z) = (0)$.

By assumption, $[\delta(\rho_{s+1})] = 1$ and there is a nonzero map $u : \rho_{s+1} \longrightarrow N$ which does not factor through the map g . This map cannot be a split monomorphism since ρ_{s+1} is not a summand of N by assumption. Therefore there is a map $u' : \rho_{s+2}^2 \oplus Z \longrightarrow N$ such that $u = u'j$. Now u' does not factor through the map g either, and $p(u')$ is a nonzero element in

$$\delta(\rho_{s+2}^2 \oplus Z) = \delta(\rho_{s+2}^2) \xrightarrow{\phi} \text{Hom}((\ , \rho_{s+2}^2), \delta),$$

where the last isomorphism is again by Yoneda's Lemma. Therefore the map

$$\phi(p(u')) = (\phi(p(u'))_1, \phi(p(u'))_2) : (\ , \rho_{s+2}^2) \longrightarrow \delta$$

is nonzero and it follows that at least one of $\phi(p(u'))_i$ is nonzero. Then the image of $\phi(p(u'))$ in δ is the unique subfunctor $\langle \rho_{s+2} \rangle$, and it contains

$$0 \neq \text{Im } \phi(p(u)) = \langle \rho_{s+1} \rangle \subseteq \langle \rho_{s+2} \rangle$$

by construction. Note that $\langle \rho_{s+1} \rangle \neq \langle \rho_{s+2} \rangle$ since the top of $\langle \rho_{s+1} \rangle$ is $F_{\rho_{s+1}}$ and the top of $\langle \rho_{s+2} \rangle$ is $F_{\rho_{s+2}}$. Hence $\langle \rho_{s+1} \rangle$ is a proper nonzero subfunctor of $\langle \rho_{s+2} \rangle$, and $\langle \rho_{s+2} \rangle$ is consequently not a simple functor. Summing up, we have

$$0 < [\langle \rho_{s+2} \rangle(\rho_{s+2})] \leq [\delta(\rho_{s+2})] = 1,$$

the top of $\langle \rho_{s+2} \rangle$ is $F_{\rho_{s+2}}$ and $\langle \rho_{s+2} \rangle$ is not simple. It follows that $F_{\rho_{s+2}}$ cannot be a summand in the socle of $\langle \rho_{s+2} \rangle$, and hence not in the socle of δ since $[\langle \rho_{s+2} \rangle(\rho_{s+2})] = [\delta(\rho_{s+2})] = 1$. Therefore ρ_s cannot be a summand of U by Proposition 2.2. \square

The second lemma we need is the following.

Lemma 5.2. *Given $0 \neq n \in \mathbb{N}$ and some Λ_q -module N such that $P_q^n <_{\text{deg}} N$, let $0 \longrightarrow U \xrightarrow{f} U \oplus P_q^n \xrightarrow{g} N \longrightarrow 0$, where g is right minimal, be a Riedtmann-sequence giving this degeneration. If there is some $x \in \mathbb{P}^1$ such that $[(x, i), P_q^n] = [(x, i), N]$ for all $i \in \mathbb{N}$, then U is a regular module.*

Proof. Recall (Subsection 3.1) that for each $s \in \mathbb{Z}$ and each $y \in \mathbb{P}^1$, there is an exact sequence

$$0 \longrightarrow \rho_s \xrightarrow{j} W \oplus \rho_{s+1} \longrightarrow (y, 1) \longrightarrow 0$$

of the form ζ_y where $W = P_q$ when $s < 0$ and $W = 0$ otherwise. Consider the exact sequence of covariant functors induced by this sequence:

$$0 \longrightarrow ((y, 1), \) \longrightarrow (W \oplus \rho_{s+1}, \) \xrightarrow{(j, \)} (\rho_s, \) \longrightarrow G_y \longrightarrow 0.$$

We have that $[G_y(\rho_t)] = 0$ for $t < s$ and that $[G_y(t)] = 1$ for $t \geq s$. It follows that for $t < s$, all maps $\rho_s \longrightarrow \rho_t$ will factor through the map j , as will all nonisomorphisms $\rho_s \longrightarrow \rho_s$. Also note that $[G_y(x, m)] = 0$ for all m when $x \neq y$, so that all maps

$\rho_s \longrightarrow (x, m)$, will also factor through j for $x \neq y$. All maps $\rho_s \longrightarrow P_q^m$, for all $m > 0$, will, of course, also factor through j since P_q is injective.

Assume that $P_q^n <_{deg} N$ and that there is some element $x \in \mathbb{P}^1$ such that $[(x, i), P_q^n] = [(x, i), N]$ for all $i \in \mathbb{N}$. Let $0 \longrightarrow U \xrightarrow{f} U \oplus P_q^n \xrightarrow{g} N \longrightarrow 0$ be a Riedtmann-sequence giving this degeneration, with the map g right minimal, and δ the contravariant functor induced by this sequence. By Corollary 2.3, (x, i) cannot be a summand in U for any i , since $[\delta(x, i)] = 0$ for all i by assumption.

Suppose there is some $s \in \mathbb{Z}$ such that ρ_s is a summand in U .

Let $s_m = \max\{s_i \mid \rho_{s_i} \text{ is a summand of } U\}$ and $U = \rho_{s_m}^r \oplus U'$, where ρ_{s_m} is not a summand of U' and $r \geq 1$. Consider the commutative diagram

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (\rho_{s_m}^{r-1} \oplus U') \oplus \rho_{s_m} & \xrightarrow{f} & U \oplus P_q^n & \xrightarrow{g} & N \longrightarrow 0 \\
& & \downarrow \begin{pmatrix} 1 & 0 \\ 0 & j \end{pmatrix} & & \downarrow & & \parallel \\
0 & \longrightarrow & (\rho_{s_m}^{r-1} \oplus U') \oplus W \oplus \rho_{s_m+1} & \longrightarrow & D & \longrightarrow & N \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & (x, 1) & \xlongequal{\quad\quad\quad} & (x, 1) & & \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array}$$

where the first column is ζ_x with $\rho_{s_m}^{r-1} \oplus U'$ added to the first and the second term with the identity map between them, the first row is the Riedtmann-sequence we started with and D is the pushout. Now U' has no summand of the form (x, i) , as we have seen; if ρ_s is a summand of U' then $s < s_m$ and no coordinate of f is an isomorphism $\rho_{s_m} \longrightarrow \rho_{s_m}$ by assumption (since g is right minimal). Moreover, P_q^n is injective. It follows that f factors through $\begin{pmatrix} 1 & 0 \\ 0 & j \end{pmatrix}$, and consequently the middle column splits and $D \simeq U \oplus P_q^n \oplus (x, 1)$.

Let δ' be the contravariant functor induced by the second row. By the 5-lemma we get induced a surjection $\delta \longrightarrow \delta'$, and so

$$\begin{aligned}
[\delta] - [\delta'] &= [, N] - [, P_q^n] - [, N \oplus W \oplus \rho_{s_m+1}] + [, \rho_{s_m} \oplus (x, 1) \oplus P_q^n] \\
&= [, \rho_{s_m} \oplus (x, 1)] - [, W \oplus \rho_{s_m+1}].
\end{aligned}$$

But $[(x, 1), \rho_{s_m} \oplus (x, 1)] - [(x, 1), W \oplus \rho_{s_m+1}] = 1$ while $[\delta(x, 1)] = 0$ by assumption, so we have a contradiction. Therefore U has no summand of the form ρ_s . \square

We have the following result on cancelling a common summand from the degenerations in Proposition 4.1.

Theorem 5.3. *Given $n \geq 2$, we have*

$$(1) \quad P_q \oplus (a, n-2) \leq_{deg} (a, n)$$

and

$$(2) \quad P_q \oplus (a, n-2) \oplus (a, n) \leq_{deg} (a, n-1) \oplus (a, n+1)$$

if and only if $a = qa$.

Proof. Assume $a \in \mathbb{P}^1$ is such that $a = qa$. Recall (Subsection 3.3) that in this case there is a sequence of the form ν_1 which for $m = n - 1$ becomes

$$0 \longrightarrow (a, n - 1) \longrightarrow (a, n - 1) \oplus P_q \oplus (a, n - 2) \longrightarrow (a, n) \longrightarrow 0.$$

It follows that $P_q \oplus (a, n - 2) \leq_{deg} (a, n)$ when $a = qa$.

It is not difficult to see that if there is a degeneration of the type (1) from the theorem, then there is a degeneration of the type (2) too: we have

$$P_q \oplus (a, n - 2) \oplus (a, n) \leq_{deg} P_q \oplus (a, n - 1) \oplus (a, n - 1) \leq_{deg} (a, n - 1) \oplus (a, n + 1)$$

where the first degeneration comes from an almost split sequence starting in $(a, n - 1)$ and the second degeneration is of type (1). Thus, $P_q \oplus (a, n - 2) \oplus (a, n)$ degenerates to $(a, n - 1) \oplus (a, n + 1)$ if $a = qa$. Hence, for $q = 1$, there are degenerations (1) and (2) for each a , while for $q \neq 1$ we have these degenerations when $a = 0$ or ∞ .

Now assume that we also have a degeneration $P_q \oplus (a, n - 2) \leq_{deg} (a, n)$ for some $a \notin \{0, \infty\}$ when $q \neq 1$ and let

$$\eta: 0 \longrightarrow U \longrightarrow U \oplus P_q \oplus (a, n - 2) \longrightarrow (a, n) \longrightarrow 0$$

be a Riedtmann-sequence giving this degeneration with the last map right minimal. By Lemma 5.1, U cannot have any summand of the form ρ_s , and so by Corollary 2.3 all summands of U must be of the form (a, n_i) . Hence, $U \simeq \bigoplus_i (a, n_i)$. Note that $\eta = \sum_i \eta_i$, where $\eta_i \in \text{Ext}^1((a, n), (a, n_i))$, and the middle term of η is a submodule of the direct sum of middle terms of η_i 's.

By Lemma 3.2, an extension of (a, n) by (a, n_i) cannot have P_q as a summand in the middle term when $a \neq qa$ (as is the case here). It follows that η cannot have P_q as a summand in the middle term either, a contradiction.

In the same way as above, using Lemma 5.1 and Lemma 3.2, one can show that $P_q \oplus (a, n - 2) \oplus (a, n) \not\leq_{deg} (a, n - 1) \oplus (a, n + 1)$ if $a \neq qa$. \square

We have the following result on cancelling of a common summand from degenerations in Proposition 4.2.

Theorem 5.4. (1) $P_q^n \leq_{deg} (a, 2n) \Leftrightarrow a = qa$;
(2) $P_q^n \leq_{deg} (a, n) \oplus (b, n) \Leftrightarrow \begin{cases} b = q^{\pm 1}a & , \quad n \text{ odd;} \\ b = q^{\pm 1}a \text{ or } (a = qa \text{ and } b = qb) & , \quad n \text{ even.} \end{cases}$

Proof. (1) If $a = qa$, recall (Subsection 3.3) that there is a sequence of the form ν_n :

$$0 \longrightarrow (a, n) \longrightarrow (a, n) \oplus P_q^n \longrightarrow (a, 2n) \longrightarrow 0.$$

Consequently, $P_q^n \leq_{deg} (a, 2n)$ if $a = qa$.

Assume now that $P_q^n \leq_{deg} (a, 2n)$ for some $a \neq qa$, i.e. $a \notin \{0, \infty\}$ when $q \neq 1$. Let

$$0 \longrightarrow U \longrightarrow U \oplus P_q^n \longrightarrow (a, 2n) \longrightarrow 0$$

be a Riedtmann-sequence giving this degeneration. We can assume that the second map is right minimal. Let δ be the contravariant functor induced by this sequence. Note that $[\delta(x, m)] \neq 0$ if and only if $x = a$. By Lemma 5.2, we get that each indecomposable summand of U is of the form (x, n) for some $x \in \mathbb{P}^1$ and $n \in \mathbb{N}$, and then by Corollary 2.3 each indecomposable summand of U is of the form (a, m_i) . Similarly as in the proof of the previous theorem, using Lemma 3.2, we get a contradiction and part (1) of the theorem is proved.

Part (2): It is easy to see that $P_q^n \leq_{deg} (a, n) \oplus (b, n)$ for $b = q^{\pm 1}a$; this degeneration is given by a sequence

$$0 \longrightarrow (q^{-1}a, n) \longrightarrow P_q^n \longrightarrow (a, n) \longrightarrow 0$$

where the second map is a projective cover. For n even, say $n = 2m$, we know from (1) that $P_q^m \leq_{deg} (a, 2m)$ if $a = qa$. It follows by the transitivity of degenerations that

$$P_q^{2m} \leq_{deg} (a, 2m) \oplus (b, 2m)$$

when $a = qa$ and $b = qb$. We want to show that these are the only possible degenerations of this type.

Assume that $b \neq q^{\pm 1}a$ and that $P_q^n \leq_{deg} (a, n) \oplus (b, n)$, i.e. there exists an exact sequence η of the form

$$\eta : 0 \longrightarrow U \longrightarrow U \oplus P_q^n \longrightarrow (a, n) \oplus (b, n) \longrightarrow 0$$

with the last map right minimal. Since $[(x, i), P_q^n] = [(x, i), (a, n) \oplus (b, n)]$ for $x \neq a, b$ we get that no indecomposable module of the form (x, i) , for $x \notin \{a, b\}$, can be a direct summand of U . Moreover, by Lemma 5.2, we get that U has no summand of the form ρ_s . It follows that $U = (\oplus_i (a, m_i)) \oplus (\oplus_j (b, m_j))$. Clearly, $\eta \in \text{Ext}^1((a, n) \oplus (b, n), U)$.

(I) Assume first that $a \neq b$. Since $b \neq q^{\pm 1}a$ by assumption, we get

$$\text{Ext}^1((a, n) \oplus (b, n), U) = \bigoplus_i \text{Ext}^1((a, n), (a, m_i)) \oplus \bigoplus_j \text{Ext}^1((b, n), (b, m_j)).$$

Therefore, we have $\eta = \eta_1 + \eta_2$, where $\eta_1 \in \bigoplus_i \text{Ext}^1((a, n), (a, m_i))$ and $\eta_2 \in \bigoplus_j \text{Ext}^1((b, n), (b, m_j))$. Let X_i be the middle term of η_i for $i = 1, 2$. Also, denote $\bigoplus_i (a, m_i)$ by A , and $\bigoplus_j (b, m_j)$ by B , so that

$$\eta_1 : 0 \longrightarrow A \longrightarrow X_1 \longrightarrow (a, n) \longrightarrow 0,$$

$$\eta_2 : 0 \longrightarrow B \longrightarrow X_2 \longrightarrow (b, n) \longrightarrow 0$$

and

$$\eta : 0 \longrightarrow \underbrace{A \oplus B}_U \longrightarrow \underbrace{X_1 \oplus X_2}_{U \oplus P_q^n} \longrightarrow (a, n) \oplus (b, n) \longrightarrow 0.$$

Assume that $n = 2m + 1$, an odd integer. Then

$$X_1 \oplus X_2 = A \oplus B \oplus P_q^{2m+1}$$

by assumption and it follows that P_q^{m+1} must be a summand of X_1 or of X_2 . Assume that it is a summand of X_2 (similarly if it is a summand of X_1), i.e. $X_2 = P_q^{m+1} \oplus X_2'$. If $l(Z)$ denotes the length of a module Z , then we have

$$l(b, n) + l(B) = l(X_2) = l(P_q^{m+1}) + l(X_2').$$

Since $l(b, n) = 2n$ and $l(P_q^{m+1}) = 2n + 2$, we get that $l(X_2') < l(B)$. Now $X_1 \oplus X_2 = A \oplus B \oplus P_q^{2m+1}$ by assumption, so there must be some summand of B , (b, m_j) , which is a summand of X_1 , i.e. $X_1 = (b, m_j) \oplus X_1'$. Write the sequence

$$\eta_1 : 0 \longrightarrow A \longrightarrow X_1 \longrightarrow (a, 1) \longrightarrow 0$$

as an exact square:

$$\begin{array}{ccc} A & \xrightarrow{\beta} & X'_1 \\ \downarrow \alpha & & \downarrow \alpha' \\ (b, m_j) & \xrightarrow{\beta'} & (a, 1) \end{array}$$

The kernel of β' must contain $\text{soc}(b, m_j)$ (since $b \neq a$) and then also the kernel of β must contain it. Moreover, since $A = \bigoplus_i (a, m_i)$, the socle of A must be contained in the kernel of α . But then $\binom{\alpha}{\beta}$ cannot be an injective map, a contradiction.

Assume now that $n = 2m$, an even integer, and $a \neq qa$ or $b \neq qb$. We still assume that $b \neq q^{\pm 1}a$. By assumption, $X_1 \oplus X_2 = A \oplus B \oplus P_q^{2m}$. As we have seen above, no summand of A can be a summand of X_2 , and no summand of B can be a summand of X_1 . Therefore we get $X_1 = A \oplus P_q^m$ and $X_2 = B \oplus P_q^m$, and it follows from the sequences η_1 and η_2 that $P_q^m \leq_{deg} (a, 2m)$ and $P_q^m \leq_{deg} (b, 2m)$, respectively. By part (1) of this theorem, it follows that $a = qa$ and $b = qb$, which contradicts our assumptions.

(II) For $q \neq 1$, assume that $a = b$. Our U from the exact sequence η is of the form $\bigoplus_i (a, m_i)$, and $\eta = \eta_1 + \eta_2$ where $\eta_1, \eta_2 \in \bigoplus_i \text{Ext}^1((a, n), (a, m_i))$. Now $\eta_1 = \sum_i \eta_{1,i}$, where

$$\eta_{1,i} : 0 \longrightarrow (a, m_i) \longrightarrow X_{1,i} \longrightarrow (a, n) \longrightarrow 0.$$

Since $a = b \neq q^{\pm 1}a$ by assumption, P_q is not a summand of $X_{1,i}$ for any i by Lemma 3.2. Since X_1 is a submodule of $\bigoplus_i X_{1,i}$, P_q is not a summand of X_1 either and similarly for X_2 . The middle term of $\eta = \eta_1 + \eta_2$ is a factor of $X_1 \oplus X_2$, and it has P_q as a summand giving a contradiction.

Therefore $P_q^n \leq_{deg} (a, n) \oplus (b, n)$ only in the situations given in the Theorem.

□

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