TOP-STABLE AND LAYER-STABLE DEGENERATIONS AND HOM-ORDER

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Abstract. Using geometrical methods, Huisgen-Zimmermann shows in [H-Z] that if M is a module with simple top, then M has no proper degeneration $M <_{deg} N$ such that $\mathfrak{r}^t M/\mathfrak{r}^{t+1} M \simeq \mathfrak{r}^t N/\mathfrak{r}^{t+1} N$ for all t. Given a module M with square-free top and a projective cover P, she shows that $\dim_k \operatorname{Hom}(M, M) = \dim_k \operatorname{Hom}(P, M)$ if and only if M has no proper degeneration $M <_{deg} N$ where $M/\mathfrak{r} M \simeq N/\mathfrak{r} N$. We prove here these results in more general form, for hom- instead of degeneration-order and we prove them algebraically. The results of Huisgen-Zimmermann follow as consequences from our results. In particular, we get that her second result holds not just for modules with square-free top, but also for indecomposable modules in general.

1. Introduction

Let Λ be a finitely generated associative k-algebra with unit where k is an algebraically closed field. By $\operatorname{mod} \Lambda$ we denote the category of finite-dimensional left Λ-modules.

A d-dimensional left Λ-module M is a d-dimensional vector space M together with a multiplication on M by Λ from the left. By choosing a basis in M one can identify M with the vector space k^d. If $\lambda_1, \lambda_2, \ldots, \lambda_r$ is a generating set for Λ over k, then the multiplication on M by λ_i induces a k-endomorphism of k^d which can be represented by a $d \times d$-matrix over k. Thus, M corresponds to a unique r-tuple $m = (m_1, m_2, \ldots, m_r)$ of $d \times d$-matrices over k such that for all polynomials f in r non-commuting variables over k with the property that $f(\lambda_1, \lambda_2, ..., \lambda_r) = 0$ in Λ, we have $f(m_1, m_2, .., m_r) = 0$ in the ring of $d \times d$-matrices over k. Conversely, each such r-tuple m corresponds to a ring homomorphism $\phi_m : \Lambda \longrightarrow \operatorname{End}_k(k^d)$ and hence gives a module structure M on k^d in the obvious way. We denote by $\operatorname{mod}^d_\Lambda(k)$ the set of all such r-tuples. It is an affine variety.

The general linear group $\operatorname{Gl}_d(k)$ acts on $\operatorname{mod}^d_\Lambda(k)$ by conjugation, $g * x = (gx_1g^{-1}, \ldots, gx_rg^{-1})$ for $g \in \operatorname{Gl}_d(k)$, $x \in \operatorname{mod}^d_\Lambda(k)$, and the orbits correspond to the isomorphism classes of d-dimensional Λ-modules (see [K]). Let M and N be two d-dimensional left Λ-modules corresponding to m and n in $\operatorname{mod}^d_\Lambda(k)$ respectively. We say that M degenerates to N if n belongs to the Zariski closure $\overline{\mathcal{O}(m)}$ of $\mathcal{O}(m)$, and we denote this by $M \leq_{deg} N$ (see [B]).

Clearly, $\leq_{deg}$ is a partial order on the set of isomorphism classes of d-dimensional modules.

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Riedtmann showed in [R] that if there is an exact sequence of the form

$$(1) \qquad 0 \longrightarrow U \longrightarrow U \oplus M \longrightarrow N \longrightarrow 0$$

for some $U \in \operatorname{mod} \Lambda$, or of the form

$$(2) \qquad 0 \longrightarrow N \longrightarrow M \oplus V \longrightarrow V \longrightarrow 0$$

for some $V \in \operatorname{mod} \Lambda$, then M degenerates to N. In particular, if

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an exact sequence, then $B \leq_{deg} A \oplus C$.

Zwara showed in [Z1] that the existence of a sequence of the form (1) is equivalent to the existence of a sequence of the form (2), and he showed in [Z2] that this is equivalent to $M \leq_{deg} N$.

From the equivalence of the degeneration order with the existence of Riedtmann-sequences above, it is easy to see that if $M \leq_{deg} N$ then

$$\dim_k \operatorname{Hom}_\Lambda(X, M) \leq \dim_k \operatorname{Hom}_\Lambda(X, N)$$

for each Λ-module X. We shall use the notation (X, Y) for $\operatorname{Hom}_\Lambda(X, Y)$ and $[X, Y]$ for $\dim_k \operatorname{Hom}_\Lambda(X, Y)$. If $[X, M] \leq [X, N]$ for each Λ-module X, we will denote this by $M \leq_{hom} N$. Auslander has proved the following.

- If $M \leq_{hom} N$ and $N \leq_{hom} M$, then M and N are isomorphic. Hence $\leq_{hom}$ is a partial order on the set of isomorphism classes of d-dimensional modules.
- $M \leq_{hom} N$ if and only if $[M, X] \leq [N, X]$ for all X.

Bongartz proved in [B] that $M <_{hom} N$ implies that the inequality $[M, M] < [N, N]$ holds. Note that the hom-order is defined also when the field k is not algebraically closed.

These two partial orders are not equivalent. Degeneration-order implies hom-order as shown above, but Carlson gave an example (see [R]) of two modules M and N such that $M \leq_{hom} N$, while $M \not\leq_{deg} N$.

2. The main results

B.Huisgen-Zimmermann defines in her preprint [H-Z] top-stable and layer-stable degenerations. She calls a degeneration $M \leq_{deg} N$ top-stable if $M/\mathfrak{r}\, M \simeq N/\mathfrak{r}\, M$ and she calls it layer stable if $\mathfrak{r}^t\, M/\mathfrak{r}^{t+1}\, M \simeq \mathfrak{r}^t\, N/\mathfrak{r}^{t+1}\, N$ for all $t = 0, \ldots, n$ where $n+1$ is the Loewy length of M.

Example. Let Λ be the Kronecker algebra which is the path algebra of the quiver $1 \rightrightarrows 2$ over k. The degeneration

$$k^2 \overset{\begin{pmatrix}1&0\\0&1\end{pmatrix}}{\underset{\begin{pmatrix}0&0\\1&0\end{pmatrix}}{\rightrightarrows}} k^2 \leq_{deg} k \overset{\binom{1}{0}}{\underset{\binom{0}{1}}{\rightrightarrows}} k^2 \;\oplus\; k \rightrightarrows 0$$

is both top-stable and layer-stable.

Example. Let Λ be the path algebra of the quiver $1 \longrightarrow 2 \longrightarrow 3$. The degeneration

$$k \xrightarrow{1} k \xrightarrow{1} k \,\oplus\, 0 \longrightarrow k \longrightarrow 0 \;\leq_{deg}\; k \xrightarrow{1} k \longrightarrow 0 \,\oplus\, 0 \longrightarrow k \xrightarrow{1} k$$

is top-stable, but not layer-stable.

Example. Given an almost split exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of Λ-modules, let $I = \mathrm{D}(\Lambda)$. If A is not a summand of $\operatorname{soc} I$, i.e A is not simple; then the degeneration $B <_{deg} A \oplus C$ is top-stable, and if A is not a summand of $\operatorname{soc}^i I$ for any i, then the degeneration $B <_{deg} A \oplus C$ is layer-stable.

We say that a module M has a square-free top if $M/\mathfrak{r}\,M = \oplus_i S_i$ where S_i's are simple modules such that $S_i \not\simeq S_j$ when $i \neq j$.

Using purely geometrical methods, Huisgen-Zimmermann shows in [H-Z] the following results.

Theorem 2.1. (a) *If M is a Λ-module with simple top, then M does not have any proper layer-stable degenerations.*

(b) *Given Λ-module M with square-free top, the following conditions are equivalent:*

1. *M does not have any proper top-stable degenerations.*
2. *$[M, M] = [P, M]$, where P is a projective cover of M.*

Using the hom-order and the algebraic characterization of degenerations by the existence of Riedtmann-sequences, these results can be proven more elegantly and in more general form, as we shall see here. But, before that, note the following.

Proposition 2.2. *Let P be a projective cover of M. Then $[M, M] \leq [P, M]$ and $[M, M] = [P, M]$ if and only if M is projective as a $\Lambda/annM$-module.*

Proof. Let

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

be an exact sequence where the last map is a projective cover of M. Applying the functor $(\ ,M)$ to this sequence, it is easy to see that $[M, M] \leq [P, M]$.

Assume that M is a projective $\Lambda/\mathrm{ann}M$-module, i.e. $M = P/\mathrm{ann}M{\cdot}P$. Applying the functor $(\ ,M)$ to the exact sequence

$$0 \longrightarrow \mathrm{ann}M \cdot P \longrightarrow P \longrightarrow P/\mathrm{ann}M \cdot P \longrightarrow 0,$$

we get the exact sequence

$$0 \longrightarrow (P/\mathrm{ann}M \cdot P, M) \longrightarrow (P, M) \longrightarrow (\mathrm{ann}M \cdot P, M).$$

The last map in this sequence is non-zero if and only if there is a map $h : P \longrightarrow M$ such that $h(\mathrm{ann}M \cdot P) \neq 0$. But, $h(\lambda x) = \lambda h(x) = 0$ for all $\lambda \in \mathrm{ann}M$ and all $x \in P$. Hence, the last map is 0 and it follows that $[P, M] = [P/\mathrm{ann}M \cdot P, M]$ for all M. By assumption, $M = P/\mathrm{ann}M \cdot P$ and hence $[M, M] = [P, M]$.

Assume now that $[M, M] = [P, M]$ and that M is not a projective $\Lambda/\mathrm{ann}M$-module. Let

$$0 \longrightarrow C \longrightarrow P/\mathrm{ann}M \cdot P \longrightarrow M \longrightarrow 0$$

be an exact sequence of $\Lambda/\mathrm{ann}M$-modules, where the last map is a projective cover. Since M is a faithful $\Lambda/\mathrm{ann}M$-module, there is a monomorphism $\Lambda/\mathrm{ann}M \longrightarrow M^r$

for some $r \in \mathbb{N}$ and hence there is a monomorphism $P/\mathrm{ann}M \cdot P \longrightarrow M^s$ for some $s \in \mathbb{N}$. It follows that there is some $u : P/\mathrm{ann}M \cdot P \longrightarrow M$ such that $u(C) \neq 0$. From above, we have that $[P/\mathrm{ann}M \cdot P, M] = [P, M]$ and hence, by assumption, $[P/\mathrm{ann}M \cdot P, M] = [M, M]$. Applying the functor $(\ , M)$ on the exact sequence above, we get then that for each map $u : P/\mathrm{ann}M \cdot P \longrightarrow M$, $u(C) = 0$. Thus, we get the wanted contradiction. □

Definition. *We define top-stable and layer-stable hom-order in a natural way: we call the hom-order $M \leq_{hom} N$ top-stable if $M/\mathfrak{r}\, M \simeq N/\mathfrak{r}\, M$ and we call it layer-stable if $\mathfrak{r}^t\, M/\mathfrak{r}^{t+1}\, M \simeq \mathfrak{r}^t\, N/\mathfrak{r}^{t+1}\, N$ for all t.*

Theorem 2.3. *Given a d-dimensional Λ-module M with simple top. If there is some d-dimensional module N such that $M \leq_{hom} N$ is layer-stable, then $M \simeq N$.*

Proof. Assume that $M \leq_{hom} N$ is layer-stable and $M/\mathfrak{r}\, M$ is a simple module, say S. We proceed by induction on the Loewy length of M to show that M and N must be isomorphic.

If the Loewy length of M is 1, the claim clearly holds. Assume that it holds for modules of Loewy length less than the Loewy length of M. Consider modules $M' = M/\mathfrak{r}^n\, M$ and $N' = N/\mathfrak{r}^n\, N$ where $n+1$ is the Loewy length of M. These two modules have the same dimension since $\mathfrak{r}^t\, M/\mathfrak{r}^{t+1}\, M \simeq \mathfrak{r}^t\, N/\mathfrak{r}^{t+1}\, N$ for all t. Moreover, we have that $\mathfrak{r}^t\, M'/\mathfrak{r}^{t+1}\, M' \simeq \mathfrak{r}^t\, N'/\mathfrak{r}^{t+1}\, N'$ for all t, and $M'/\mathfrak{r}\, M' \simeq S$. It is not difficult to see that $M/\mathfrak{r}^n\, M \leq_{hom} N/\mathfrak{r}^n\, N$ as we shall see here. Given a Λ-module X with Loewy length less than $n+1$, the Loewy length of M, we have inequality

$$[M/\mathfrak{r}^n\, M, X] = [M, X] \leq [N, X] = [N/\mathfrak{r}^n\, N, X]$$

since $M \leq_{hom} N$. If X is a Λ-module with Loewy length greater than or equal to $n+1$, then $[M/\mathfrak{r}^n\, M, X] = [M/\mathfrak{r}^n\, M, \mathrm{soc}^n\, X]$ (and $[N/\mathfrak{r}^n\, N, X] = [N/\mathfrak{r}^n\, N, \mathrm{soc}^n\, X]$) and $\mathrm{soc}^n\, X$ has Loewy length n. Therefore, by the previous remark, we will again have $[M/\mathfrak{r}^n\, M, X] \leq [N/\mathfrak{r}^n\, N, X]$. Thus we have this inequality for each module X and consequently $M/\mathfrak{r}^n\, M \leq_{hom} N/\mathfrak{r}^n\, N$. By induction, it follows that $M/\mathfrak{r}^n\, M \simeq N/\mathfrak{r}^n\, N$.

We have that $M \leq_{hom} N$ by assumption. In particular, we must have $[M, M] \leq [N, M]$. Assume that M and N are not isomorphic. We have then equalities

$$[M, M] - 1 = [M, \mathfrak{r}\, M] \quad \text{and} \quad [N, M] = [N, \mathfrak{r}\, M]$$

since $M/\mathfrak{r}\, M$ is a simple module. Moreover, we have that

$$[M, \mathfrak{r}\, M] = [M/\mathfrak{r}^n\, M, \mathfrak{r}\, M] \quad \text{and} \quad [N, \mathfrak{r}\, M] = [N/\mathfrak{r}^n\, N, \mathfrak{r}\, M].$$

Since $M/\mathfrak{r}^n\, M \simeq N/\mathfrak{r}^n\, N$, we get that $[M/\mathfrak{r}^n\, M, \mathfrak{r}\, M] = [N/\mathfrak{r}^n\, N, \mathfrak{r}\, M]$ and it follows that $[M, M] > [N, M]$, a contradiction. Therefore M and N must be isomorphic. □

Since the degeneration-order implies the hom-order between two modules, we get the result (a) of Theorem 2.1 as a consequence.

Note that the Theorem 2.3 above holds also when k is not algebraically closed. The only difference we have to make in the proof is that one have to observe that $[M, M] - a = [M, \mathfrak{r}\, M]$ where $a \geq 1$.

Theorem 2.4. *Let M be a d-dimensional Λ-module and let P be its projective cover.*

(1) *If $[M,M]=[P,M]$ and N is a d-dimensional module such that $M \leq_{hom} N$ is top-stable, then $M \simeq N$.*
(2) *If M is an indecomposable module or has a square-free top and $[M,M] < [P,M]$, then M has a proper top-stable degeneration.*

Proof. (1) Assume that $[M,M]=[P,M]$ and that there is some d-dimensional module N such that $M <_{hom} N$ is top-stable. Since $M/\mathfrak{r}\,M \simeq N/\mathfrak{r}\,N$, we know that P is a projective cover of N too. It follows by Proposition 2.2 that $[N,N] \leq [P,N]$. Since M and N have the same composition factors by assumption, $[P,N]=[P,M]$, hence $[N,N] \leq [M,M]$ which is a contradiction since $[M,M] < [N,N]$ when $M <_{hom} N$.

(2) Assume first that M is an indecomposable module such that $[M,M] < [P,M]$. Let

$$\eta: \quad 0 \longrightarrow K \xrightarrow{f} P \xrightarrow{g} M \longrightarrow 0$$

be an exact sequence where the map g is a projective cover of M. Since $[M,M] < [P,M]$ by assumption, we get that there is a map $h: P \longrightarrow M$ such that $hf \neq 0$. Moreover, we can assume that the map h is not an epimorphism, as we shall see here. Assume that h is an epimorphism, and denote by π the natural epimorphism from M to $M/\mathfrak{r}\,M$. Then both πg and πh are epimorphisms from P to $M/\mathfrak{r}\,M$, and they induce isomorphisms

$$\overline{\pi g} \text{ and } \overline{\pi h}: P/\mathfrak{r}\,P \longrightarrow M/\mathfrak{r}\,M, \text{ respectively.}$$

Since k is an algebraically closed field, $\operatorname{Hom}(S_i,S_j) \simeq k$ if $S_i \simeq S_j$ and $\operatorname{Hom}(S_i,S_j)=0$ otherwise. Hence, the maps $\overline{\pi g}$ and $\overline{\pi h}$ are given by $r \times r$-matrices A and B over k, respectively, where r is the number of indecomposable summands of the top of M (and P). By changing the basis if necessary, we can assume that $\overline{\pi g}$ is the identity, i.e. $A = I$, the identity matrix. We want to find some $\alpha \in k$ such that $h+\alpha g$ is not an epimorphism. In other words, we want $\alpha \in k$ such that $\det(B+\alpha I)=0$. Let $a \in k$ be one of eigenvalues of B and let $\alpha = -a$. Clearly, $\det(B+\alpha I)=0$ and $h+\alpha g$ is not an epimorphism. Also, $(h+\alpha g)f = hf \neq 0$ and therefore, if h is an epimorphism, we can use $h+\alpha g$ instead.

Since P is a projective module, there is a map $g': P \longrightarrow P$ such that $gg' = h$. Let $K_0 = \operatorname{Ker} g'$ and assume that $K \cap K_0 = (0)$. Then $\binom{g'}{g}: P \longrightarrow P \oplus M$ is a monomorphism and it is not split since M is indecomposable and h is not an epimorphism. Consequently, $M <_{deg} \operatorname{Coker}\binom{g'}{g} = N$ since we get a (non-split) Riedtmann-sequence

$$0 \longrightarrow P \xrightarrow{\binom{g'}{g}} P \oplus M \xrightarrow{(u\ v)} N \longrightarrow 0.$$

Note that u is an epimorphism (since g is) and also $\operatorname{Ker} u \simeq \operatorname{Ker} g \subseteq \mathfrak{r}\,P$. Hence M and N have isomorphic tops and the degeneration is top-stable.

Assume now that $K \cap K_0 \neq (0)$. Since $hf \neq 0$, $K \cap K_0$ is a proper submodule of K. Let

$$\pi_0: P \longrightarrow P/K \cap K_0$$

be the natural epimorphism and let $g_0 : P/K \cap K_0 \longrightarrow M$ be such that $g = g_0\pi_0$. We get induced exact sequence

$$\eta_0 : \quad 0 \longrightarrow K/K \cap K_0 \xrightarrow{f_0} P/K \cap K_0 \xrightarrow{g_0} M \longrightarrow 0.$$

For $i > 0$, let

$$\pi_i : P \longrightarrow P/K \cap K_i$$

be the natural epimorphism where $K_i := \operatorname{Ker}\phi_{i-1}$ and $\phi_{i-1} := \pi_{i-1}g'$. By the construction, we get that $K_0 \subseteq K_1 \subseteq K_2 \subseteq \ldots$, hence

$$K \cap K_0 \subseteq K \cap K_1 \subseteq K \cap K_2 \subseteq \ldots.$$

Since $hf \neq 0$, $K \cap K_i$ is a proper submodule of K and hence $K/K \cap K_i \neq 0$ for each i.

Since $K \cap K_0$ is noetherian, there must be some t such that $K \cap K_t = K \cap K_{t+1}$. Consider the exact sequence η_t:

$$\eta_t : \quad 0 \longrightarrow K/K \cap K_t \xrightarrow{f_t} P/K \cap K_t \xrightarrow{g_t} M \longrightarrow 0.$$

By the definition, $\phi_t = \pi_t g' : P \longrightarrow P/K \cap K_t$ is such that $g_t\phi_t = h$ and $\operatorname{Ker}\phi_t = K_{t+1}$. This map induces a map $\overline{\phi_t} : P/K \cap K_{t+1} = P/K \cap K_t \longrightarrow P/K \cap K_t$. Consider the map

$$\begin{pmatrix} \overline{\phi_t} \\ g_t \end{pmatrix} : P/K \cap K_t \longrightarrow (P/K \cap K_t) \oplus M.$$

This map is a monomorphism since

$$(\operatorname{Ker} g_t) \cap (\operatorname{Ker} \overline{\phi_t}) = (K/K \cap K_t) \cap (K_{t+1}/K \cap K_t) = (0).$$

Now, M is indecomposable by assumption and $\overline{\phi_t}$ is not an epimorphism since h is not. It follows that the map $\binom{\overline{\phi_t}}{g_t}$ is not split. Let $N = \operatorname{Coker}\binom{\overline{\phi_t}}{g_t}$. We have a non-split exact sequence

$$0 \longrightarrow P/K \cap K_t \xrightarrow{\binom{\overline{\phi_t}}{g_t}} (P/K \cap K_t) \oplus M \xrightarrow{(u,v)} N \longrightarrow 0$$

and hence $M <_{deg} N$. Also, u is an epimorphism since g_t is and $\operatorname{Ker} u \simeq \operatorname{Ker} g_t \subseteq \mathfrak{r}(P/K \cap K_t)$. Hence, M and N have isomorphic tops and we are done.

Assume now that M is, possibly decomposable, module with square-free top and $[M, M] < [P, M]$, and let us show that then there is some proper top-stable degeneration of M. Let $M = \oplus M_i$ where M_i is an indecomposable module for each i. If there is some j such that the top of M_j is not simple, then, clearly, M_j is not a projective $\Lambda/\text{ann}M_j$-module and it has a proper top-stable degeneration N_j as we have seen above. But then the degeneration

$$M = \oplus_{i=1}^{r} M_i <_{deg} M_1 \oplus \ldots \oplus M_{j-1} \oplus N_j \oplus M_{j+1} \oplus \ldots M_r$$

is proper and top-stable and we are done.

So, assume that $M = \oplus_i M_i$, where M_i is an indecomposable module with simple top for each i, i.e. M is a direct sum of local modules. To prove that there is some proper top-stable degeneration of M, we can proceed as in the case of M being indecomposable above. But, to be sure that the Riedtmann-sequence arising is not split, we need some stronger condition on the map $h : P \longrightarrow M$. Before it was enough that h is not an epimorphism, now we need that $\operatorname{Im} h \subseteq \mathfrak{r} M$.

Assume that $\operatorname{Im} h \not\subseteq \mathfrak{r} M$. Let g be, as before, a projective cover of M and $\pi : M \longrightarrow M/\mathfrak{r} M$ a natural epimorphism. The maps $\pi h,\ \pi g : P \longrightarrow M/\mathfrak{r} M$ are then non-zero and they induce maps

$$\overline{\pi h}, \overline{\pi g} : P/\mathfrak{r} P \longrightarrow M/\mathfrak{r} M,$$

respectively. Since k is algebraically closed field, for two simple modules S and S' we have that $\operatorname{Hom}(S, S') \simeq k$ if $S \simeq S'$ and $\operatorname{Hom}(S, S') = 0$ otherwise. Thus, we can represent maps $\overline{\pi h}, \overline{\pi g}$ by two $r \times r$-matrices over k, A and B respectively, where r is the number of indecomposable summand of $M/\mathfrak{r} M$. Now, $\overline{\pi g}$ is an isomorphism and, changing the basis if necessary, we can assume that $B = I$, the identity matrix. Since the top of M is square-free, the matrix A is a diagonal matrix having $a_{11}, \ldots, a_{rr} \in k$ on the diagonal.

Let $\phi : M/\mathfrak{r} M \longrightarrow M/\mathfrak{r} M$ be given by the matrix $-A$ and let $\phi' : M = \oplus_i M_i \longrightarrow M = \oplus_i M_i$ be the map induced by ϕ, i.e.

$$\phi'(m_1, \ldots, m_r) = (-a_{11} m_1, \ldots, -a_{rr} m_r)$$

(ϕ' is well-defined since M_i is a local module for each i). Since $\pi(h + \phi' g) = 0$, it follows that $\operatorname{Im}(h + \phi' g) \subseteq \mathfrak{r} M$. Also, $(h + \phi' g) f = hf \neq 0$. Hence, if $\operatorname{Im} h \not\subseteq \mathfrak{r} M$, there is some $\phi' : M \longrightarrow M$ such that $\operatorname{Im}(h + \phi' g) \subseteq \mathfrak{r} M$ and $(h + \phi' g) f = hf \neq 0$. Hence we can use the map $h + \phi' g$ instead of h. $\square$

Thus, by the theorem above, we get that the result (b) from Theorem 2.1 holds not just for modules with square-free top, but also for indecomposable modules.

The part (2) of Theorem 2.4 does not hold for an arbitrary module M. Here is a counterexample.

Example. Given the k-algebra $\Lambda = k[x]/(x^2)$, let $M = \Lambda \oplus S$, where S is the simple Λ-module. A projective cover of M is $P = \Lambda^2$ and $5 = [M, M] < [P, M] = 6$. But, it is easy to see that there is no 3-dimensional module N such that $M <_{hom} N$ (or/and $M <_{deg} N$) is top-stable.

Note that the part (1) of the theorem above holds also when the field k is not algebraically closed.

By the part (2) of the theorem we get that if M is an indecomposable module or has a square-free top and $[M, M] < [P, M]$, then there is a d-dimensional module N such that $M <_{hom} N$ is top-stable. This does not hold if the field k is not algebraically closed, as the following example shows.

Example. Given the $\mathbb{R}$-algebra $\Lambda = \left(\begin{smallmatrix} \mathbb{C} & 0 \\ \mathbb{C} & \mathbb{R} \end{smallmatrix}\right)$. See [ARS], III.2, for the following: Let $\mathcal{C}$ be the category whose objects are triples (A, B, f), where $A \in \operatorname{mod} \mathbb{C}$, $B \in \operatorname{mod} \mathbb{R}$ and $f : A \longrightarrow B$ is an $\mathbb{R}$-morphism, and the morphisms between two objects (A, B, f) and (A', B', f') are pairs of morphisms (α, β) where $\alpha : A \longrightarrow A'$ ia a $\mathbb{C}$-morphism, $\beta : B \longrightarrow B'$ is an $\mathbb{R}$-morphism and $f'\alpha = \beta f$. There is an equivalence

$$F : \operatorname{mod} \Lambda \longrightarrow \mathcal{C}.$$

Let P be a Λ-module given by the triple $(\mathbb{C}, \mathbb{C}, 1_{\mathbb{C}})$ and let K be a module given by the triple $(0, \mathbb{R}, 0)$; P is an indecomposable projective module and K is a simple module. Given $\beta : \mathbb{R} \longrightarrow \mathbb{C}$ by $\beta(x) = x$, it is easy to see that the map $(0, \beta)$ gives a monomorphism from K to P. Let M be a Λ-module given by $(\mathbb{C}, \mathbb{C}, 1_{\mathbb{C}}) / \operatorname{Im}(0, \beta)$, i.e. given by the triple $(\mathbb{C}, \mathbb{R}, f)$ where $f : \mathbb{C} \longrightarrow \mathbb{R}$ is given by $f(x + yi) = y$. Now, P is a projective cover of M and

$$1 = [M, M] < [P, M] = 2.$$

Note that M is a 3-dimensional module with simple top given by the triple $(\mathbb{C}, 0, 0)$ and simple radical given by the triple $(0, \mathbb{R}, 0)$; M is an injective envelope of $(0, \mathbb{R}, 0)$. Let N be a Λ-module which is 3-dimensional as an $\mathbb{R}$-space with top $(\mathbb{C}, 0, 0)$. It follows that N must be given by a triple $(\mathbb{C}, \mathbb{R}, g)$ where $g : \mathbb{C} \longrightarrow \mathbb{R}$ is non-zero, i.e g is given by $g(x + yi) = ax + by$ for $(a, b) \neq (0, 0)$ in $\mathbb{R} \times \mathbb{R}$. The map $(\alpha, 1_{\mathbb{R}})$, where $\alpha : \mathbb{C} \longrightarrow \mathbb{C}$ is given by $\alpha(x + yi) = (x + yi)(b + ai)$, gives an isomorphism between N and M. In other words, if M and N have isomorphic tops, then they are isomorphic.

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