

2-categorical K -theories

Nils A. Baas*, baas@math.ntnu.no,
Marcel Bökstedt, marcel@imf.au.dk,
Tore August Kro†, toreak@math.ntnu.no

23rd November 2006

Abstract

For 2-categories $2\mathcal{C}$ we define a notion of associated charted $2\mathcal{C}$ -bundles and $2K$ -theories. For the 2-category of 2-vector spaces in the sense of M.M. Kapranov and V.A. Voevodsky this gives the $2K$ -theory of N.A. Baas, B.I. Dundas and J. Rognes. When the 2-category is good, we prove that its 2-nerve is a classifying space for the associated charted $2\mathcal{C}$ -bundles. In the process of proving this result we develop a lot of powerful machinery which may be useful in further studies of 2-categorical topology. As a corollary we get a new proof of the classification of principal bundles. As another corollary it follows that if we take the 2-category to be the 2-vector spaces introduced by J.C. Baez and A.S. Crans, then the associated $2K$ -theory is just two copies of ordinary K -theory.

1 Introduction.

This paper is concerned with theoretical results on topological 2-categories, and applications of theory this theory to 2-vector bundles.

The bulk of the paper discusses the theory of bundles corresponding to a topological 2-category. The intuition behind this is as follows. Transition functors of a bundle on a space X should be given by transition data on a covering of X . Since we are interested in topological 2-categories, we insist that these data are given by continuous functions. A refinement of the given covering should give rise to data on the refinement, and the resulting 2-bundle should be considered equivalent to the original one. Finally, we want concordant 2-bundle data to give equivalent 2-bundles. We formalize this in

*Support by the Institut Mittag-Leffler (Djursholm, Sweden) is gratefully acknowledged.

†Support by the Institut Mittag-Leffler (Djursholm, Sweden) is gratefully acknowledged.

section 2 as the concept of charted $2\mathcal{C}$ -bundles. For each 2-category $2\mathcal{C}$ and each space X we obtain a set of concordance classes of charted $2\mathcal{C}$ -bundles. This set is functorial in both $2\mathcal{C}$ and in X in the usual way.

A discrete 2-category has a 2-nerve (or geometric nerve), which is a simplicial set, [Dus02]. There are obvious generalizations of these nerves to topological 2-categories, which produce simplicial spaces. A charted 2-bundle is the same as a simplicial map from the simplicial space given by a covering to the geometric nerve of the 2-category.

We want to show that homotopy classes of maps from X to the realization of this topological space corresponds bijectively to concordance classes of charted 2-bundles. That is, we want to show that the realization of the topological nerve is a classifying space for the charted 2-bundles. In the classical 1-category situation, this is true for instance for the special case of topological groupoids. In analogy, we would like to have that the nerve is a classifying space at least for topological 2-groupoids, whatever that term would mean.

In section 3 we discuss the concept of a classifying space. We emphasize that in order to name something a classifying space, one should know what this space classifies. We show that the nerve of a good topological 2-category is indeed a classifying space for charted 2-bundles.

The theory of the first three sections is used in section 4 to study the 2-vector spaces defined by J.C. Baez and A.S. Crans in [BC04]. This corresponds to the concordance classes of charted bundles for a nice topological 2-groupoid, so it is determined by the realization of its nerve. We show that this nerve is homotopy equivalent to the product of two copies of the classifying space of ordinary K -theory, so the concordance classes of these 2-vector bundles are the same as pairs of concordance classes of ordinary vector bundles.

This can be seen as a negative result, in the sense that these 2-vector bundles are not related to elliptic cohomology. Clearly there may be other interesting 2-categories to consider together with their associated $2K$ -theories in the above sense. Therefore we think that the general setting developed here may be useful in such studies. In particular, there might exist a 2-groupoid giving a $2K$ -theory of chromatic filtration 2.

We point out that N.A. Baas, B.I. Dundas and J. Rognes have introduced an alternative concept of 2-vector bundles [BDR04], based on M.M. Kapranov and V.A. Voevodsky's notion of 2-vector space [KV94]. The associated $2K$ -theory is represented by the infinite loop space $K(ku)$, which is of essential chromatic filtration 2, as is elliptic cohomology.

The last five sections are more technical. They deal with facts which are used in the proofs of the statements of section 3. In section 5 we study the following problem. Suppose that Z_\bullet is a simplicial space. By applying the simplicial functor degreewise, we get a bisimplicial set. Let us diagonalize this bisimplicial set. What are the reasonable conditions on Z_\bullet that

ensure that this diagonal simplicial set has the Kan property? The answer to this question (theorem 5.3) is then used to study the geometric nerve of a topological 2-groupoid.

In section 6 we provide criteria for what we think a good and sufficiently fibrant topological 2-groupoid should be. The main criteria for sufficiently fibrant are that the source and target maps should be Serre fibrations for both 1-morphism and 2-morphisms, and that all structure used to define the 2-groupoid structure (including the equivalence of category structure) should be given by continuous maps.

In section 7 we generalize the concept of concordance classes by replacing the geometric nerve of a topological category with an arbitrary simplicial space. We give some general properties (homotopy invariance, exact sequence, gluing) for such generalized concordance classes.

In section 8 we assume that Z_\bullet is a simplicial space that satisfies the relevant Kan condition. In theorem 8.1 we show that for such simplicial spaces, concordance classes of maps from the topological space X into Z_\bullet are classified by homotopy classes of maps from X to the realization $|Z_\bullet|$.

In section 9 we finish the proof of theorem 8.1. In particular we use Quillen's small object argument to eliminate the assumption that Z_\bullet has to satisfy the topological Kan condition.

By a *2-category* we mean what is sometimes called a bicategory [Bén67], or a weak 2-category, i.e. where the associativity conditions of the composition is relaxed. We specify strict 2-categories when we need them.

2 Charted $2\mathcal{C}$ -bundles

In this section we formulate equivalent ways to define a charted $2\mathcal{C}$ -bundle. Here $2\mathcal{C}$ will be a topological 2-category. Also the objects are allowed to form topological spaces. This includes discrete (or ordinary) 2-categories as a special case, i.e. when the spaces of objects, 1- and 2-morphisms have the discrete topology.

We start with an explicit definition which specializes to the charted 2-vector bundles of [BDR04] for a suitably chosen $2\mathcal{C}$. An *ordered open cover* $(\mathcal{U}, \mathcal{I})$ of a topological space X , as defined in [BDR04], is a cover \mathcal{U} of X indexed by a partially ordered set \mathcal{I} that restricts to a total ordering on each finite subset $\{\alpha_0, \dots, \alpha_k\}$ of \mathcal{I} whenever $U_{\alpha_0 \dots \alpha_k} = U_{\alpha_0} \cap \dots \cap U_{\alpha_k}$ is nonempty. A cover \mathcal{U} is *good* if each finite intersection $U_{\alpha_1 \dots \alpha_k}$ is either empty or contractible.

Definition 2.1 *Let X be a topological space, and let $2\mathcal{C}$ be a topological 2-category. A charted $2\mathcal{C}$ -bundle \mathcal{E} over X consists of an ordered open cover $(\mathcal{U}, \mathcal{I})$ together with*

- (1) *for each α in \mathcal{I} a continuous map $V^\alpha : U_\alpha \rightarrow \text{Obj}(2\mathcal{C})$, and*

(2) for each $\alpha < \beta$ a continuous family of 1-morphisms in $2\mathcal{C}$

$$E^{\alpha\beta} : V^\alpha \rightarrow V^\beta \quad ,$$

and

(3) for each $\alpha < \beta < \gamma$ a continuous family of 2-morphisms in $2\mathcal{C}$

$$\phi^{\alpha\beta\gamma} : E^{\alpha\gamma} \Rightarrow E^{\beta\gamma} * E^{\alpha\beta} \quad ,$$

such that

(4) the diagram

$$\begin{array}{ccc} E^{\gamma\delta} * (E^{\beta\gamma} * E^{\alpha\beta}) & \xrightarrow{\underline{\alpha}} & (E^{\gamma\delta} * E^{\beta\gamma}) * E^{\alpha\beta} \\ \uparrow \phi^{\beta\gamma\delta} * 1 & & 1 * \phi^{\alpha\beta\gamma} \uparrow \\ E^{\beta\delta} * E^{\alpha\beta} & \xleftarrow{\phi^{\alpha\beta\delta}} E^{\alpha\delta} \xrightarrow{\phi^{\alpha\gamma\delta}} & E^{\gamma\delta} * E^{\alpha\gamma} \end{array}$$

commutes over $U_{\alpha\beta\gamma\delta}$, for each chain $\alpha < \beta < \gamma < \delta$ in \mathcal{I} . The 2-morphism $\underline{\alpha}$ is the natural associativity isomorphism of the 2-category $2\mathcal{C}$.

As we shall see in the following examples, it is often possible to interpret V^α , $E^{\alpha\beta}$, and $\phi^{\alpha\beta\gamma}$ as bundles of objects, 1-morphisms, and 2-morphisms respectively.

Example 2.2 Recall from [MS00, section 3.1] the notion of a local bundle gerbe. This fits into our framework as follows. Let $2\mathcal{G}$ be the topological 2-category with $\text{Obj}(2\mathcal{G}) = *$, $1 \text{ Mor}(2\mathcal{G}) = \mathbb{C}\mathbb{P}^\infty$, and interpreting $s, t \in \mathbb{C}\mathbb{P}^\infty$ as complex lines in \mathbb{C}^∞ , a 2-morphism $\phi : s \Rightarrow t$ is the same as a linear isomorphism between these lines. Given lines $l = [z_1, z_2, z_3, \dots]$ and $l' = [z'_1, z'_2, z'_3, \dots]$ their tensor product is defined as the collection of all products $z_i z'_j$, i.e. $l \otimes l' = [z_1 z'_1, z_2 z'_1, z_1 z'_2, z_2 z'_2, z_3 z'_1, \dots]$. This gives a map $\mathbb{C}\mathbb{P}^\infty \times \mathbb{C}\mathbb{P}^\infty \rightarrow \mathbb{C}\mathbb{P}^\infty$, and we use this as the composition of 1-morphisms in $2\mathcal{G}$. Recall that $\mathbb{C}\mathbb{P}^\infty$ classifies complex line bundles. Hence our $E^{\alpha\beta}$'s are complex line bundles over $U_{\alpha\beta}$. Moreover, each $\phi^{\alpha\beta\gamma}$ is a bundle isomorphism between $E^{\alpha\gamma}$ and $E^{\beta\gamma} \otimes E^{\alpha\beta}$. These correspond to the trivializations $\theta_{\alpha\beta\gamma}$ in [MS00].

Example 2.3 Based on the 2-vector spaces of [KV94] we define a topological 2-category $2\mathcal{KV}_{pt}(n)$ as follows: Set $\text{Obj}(2\mathcal{KV}_{pt}(n)) = *$, and let the 1-morphisms be the space of $n \times n$ -matrices of complex Grassmanians

$$\coprod (\text{Gr}(a_{ij}))_{i,j=1}^n$$

where the disjoint union runs over all invertible $n \times n$ -matrices (a_{ij}) consisting of non-negative integers. Thus a map $E^{\alpha\beta} : U_{\alpha\beta} \rightarrow 1 \text{Mor}(2\mathcal{KV}_{\text{pt}}(n))$ is an $n \times n$ -matrix $(E_{ij}^{\alpha\beta})$ of complex vector bundles such that $\det(\dim E_{ij}^{\alpha\beta}) = \pm 1$. The 2-morphisms are defined as matrices of linear isomorphisms, and the composition, $*$, of 1-morphisms is defined as reverse matrix multiplication using the exterior \otimes of Grassmannians as product between matrix entries, and \oplus as sum. This recovers the definition of charted 2-vector bundles given in [BDR04]. Their composition \cdot is dual to our $*$, and our $\phi^{\alpha\beta\gamma}$ is the inverse of theirs.

Example 2.4 (String bundles) Let $2\mathcal{S}$ be Tillmann's 2-category of closed strings, oriented cobordisms and their diffeomorphisms. It is easy to show that $\text{Con}(X, \Delta 2\mathcal{S}) \cong \text{Con}(X, N \bullet \mathcal{C}_2)$, where \mathcal{C}_2 is Madsen and Weiss' topological oriented cobordism category. Moreover, they show that $|N \bullet \mathcal{C}_2| \simeq \Omega^{\infty-1} \mathbb{C}P_{-1}^{\infty}$. Hence, this space classifies string bundles.

To an ordered cover $(\mathcal{U}, \mathcal{I})$ of X we associate the Čech complex \check{C}_{\bullet} , the ordered Čech complex U_{\bullet} and Segal's topological category $X_{\mathcal{U}}$. These are defined as follows: \check{C}_{\bullet} and U_{\bullet} are 1-coskeletal simplicial spaces (and thus determined by their 0- and 1-simplices, see [Bek04, proposition 3.1]). We define \check{C}_0 and \check{C}_1 to be

$$\coprod_{\alpha} U_{\alpha} \quad \text{and} \quad \coprod_{\alpha, \beta} U_{\alpha\beta}$$

respectively. U_{\bullet} is the sub-simplicial space of \check{C}_{\bullet} having the same 0-simplices, but the 1-simplices being the disjoint union of all $U_{\alpha\beta}$'s with $\alpha \leq \beta$. The space of objects of $X_{\mathcal{U}}$ is the disjoint union of all U_S where S is a finite set of indices. The space of morphisms is the disjoint union of U_S over all pairs $R \subseteq S$, with S as above. The source map forgets R , and the target map is induced from the inclusions $U_S \subseteq U_R$.

From [DI04, theorem 2.1, proposition 2.6 and proposition 2.7] we have that the spaces $|\check{C}_{\bullet}|$, $|U_{\bullet}|$ and $|N \bullet X_{\mathcal{U}}|$ all are weakly equivalent to X via the natural maps. The bars $|\ - |$ denote geometric realization.

Recall from [Str87], [BC03], or [Dus02], the notion of the geometric nerve $\Delta 2\mathcal{C}$ of $2\mathcal{C}$. It is well-defined for discrete 2-categories. $\Delta 2\mathcal{C}$ has 0-simplices the objects x_0 of $2\mathcal{C}$. The 1-simplices are the 1-morphisms $x_0 \xrightarrow{x_{01}} x_1$ of $2\mathcal{C}$, and the 2-simplices are triangles

$$\begin{array}{ccc} & x_1 & \\ x_{01} \nearrow & \uparrow & \searrow x_{12} \\ x_0 & \xrightarrow{x_{02}} & x_2 \end{array}$$

x_{012}

where x_{012} is a 2-morphism $x_{02} \Rightarrow x_{12} * x_{01}$. For $n \geq 3$ the n -simplices are built from 2-simplices, such that for each subtetrahedron the obvious

coherence condition for 2-morphisms is satisfied. The geometric nerve is 3-coskeletal. The definition extends to the cases where $2\mathcal{C}$ is topological or simplicial, and then $\Delta 2\mathcal{C}$ becomes a simplicial space or a bisimplicial set respectively.

Definition 2.5 *To an ordered cover $(\mathcal{U}, \mathcal{I})$ of X we can also associate a topological 2-category $2X_{\mathcal{U}}$. Let the space of objects be the disjoint union $\coprod U_{\alpha}$, and we denote an object by x_{α} where $x \in U_{\alpha}$. We define a space of generators for the 1-morphisms to be the disjoint union $\coprod_{\alpha < \beta} U_{\alpha\beta}$, and we denote such generating 1-morphisms by*

$$x^{\alpha\beta} : x_{\alpha} \rightarrow x_{\beta} \quad .$$

Let $\widetilde{2X_{\mathcal{U}}}$ be the smallest topological 2-category having objects as above, and containing the space of generators inside its 1-morphisms, and such that continuous strict 2-functors from $\widetilde{2X_{\mathcal{U}}}$ into an arbitrary topological 2-category $2\mathcal{C}$ are in one-to-one correspondence with commutative diagrams

$$\begin{array}{ccc} \coprod U_{\alpha} & \longrightarrow & \text{Obj } 2\mathcal{C} \\ \text{source} \uparrow & & \uparrow \text{source} \\ \coprod_{\alpha < \beta} U_{\alpha\beta} & \longrightarrow & 1 \text{ Mor } 2\mathcal{C} \quad . \\ \text{target} \downarrow & & \downarrow \text{target} \\ \coprod U_{\alpha} & \longrightarrow & \text{Obj } 2\mathcal{C} \end{array}$$

In $\widetilde{2X_{\mathcal{U}}}$ the 1-morphisms are parenthesized words of generators, whereas the 2-morphisms consist only of the associators switching the parenthesizing. For example if $\alpha < \beta < \gamma < \delta < \epsilon$ and $x \in U_{\alpha\beta\gamma\delta\epsilon}$, then all the following expressions are different 1-morphisms:

$$\begin{aligned} x^{\alpha\beta}, \quad x^{\alpha\gamma}, \quad x^{\beta\gamma} * x^{\alpha\beta}, \quad (x^{\gamma\delta} * x^{\beta\gamma}) * x^{\alpha\beta}, \\ x^{\gamma\delta} * (x^{\beta\gamma} * x^{\alpha\beta}), \quad (x^{\delta\epsilon} * x^{\gamma\delta}) * (x^{\beta\gamma} * x^{\alpha\beta}) \quad . \end{aligned}$$

*Observe that any 1-morphism $f : x_{\alpha_0} \rightarrow x_{\alpha_k}$ in $\widetilde{2X_{\mathcal{U}}}$ has an associated chain $\alpha_0 < \alpha_1 < \dots < \alpha_k$ in \mathcal{I} , i.e. f is a parenthesizing of $x^{\alpha_{k-1}\alpha_k} * \dots * x^{\alpha_1\alpha_2} * x^{\alpha_0\alpha_1}$ for some $x \in U_{\alpha_0\dots\alpha_k}$.*

Now define $2X_{\mathcal{U}}$ to be the topological 2-category with the same objects and 1-morphisms as $\widetilde{2X_{\mathcal{U}}}$, but having additional 2-morphisms. Suppose we have two 1-morphisms $f : x_{\alpha_0} \rightarrow x_{\alpha_k}$ and $g : y_{\beta_0} \rightarrow y_{\beta_l}$. Define the space of 2-morphisms $f \Rightarrow g$ in $2X_{\mathcal{U}}$ to be empty unless

$$x = y \text{ in } X,$$

$$\alpha_0 = \beta_0 \text{ and } \alpha_k = \beta_k, \text{ and}$$

the chain $\alpha_0 < \alpha_1 < \dots < \alpha_k$ is a refinement of $\beta_0 < \beta_1 < \dots < \beta_k$.

In that case $2\text{Mor}(f, g)$ is a single point. We denote the unique 2-morphism $x^{\alpha\gamma} \Rightarrow x^{\beta\gamma} * x^{\alpha\beta}$ by $x^{\alpha\beta\gamma}$. Observe that any 2-morphism in $2X_{\mathcal{U}}$ can be built, using vertical and horizontal composition, from associators and forgetting the middle element 2-morphisms, $x^{\alpha\beta\gamma}$.

Theorem 2.6 Let $(\mathcal{U}, \mathcal{I})$ be an ordered open cover over X . Then there are one-to-one correspondences between

charted $2\mathcal{C}$ -bundles over X subordinate to the given ordered open cover,

continuous strict 2-functors $2X_{\mathcal{U}} \rightarrow 2\mathcal{C}$, and

simplicial maps $U_{\bullet} \rightarrow \Delta 2\mathcal{C}$.

Proof: The proof is to inspect the definitions. Assume that $\alpha < \beta < \gamma < \delta$ in \mathcal{I} . From a strict 2-functor F we recover V^{α} , $E^{\alpha\beta}$ and $\phi^{\alpha\beta\gamma}$ of the charted $2\mathcal{C}$ -bundle at a point x in U_{α} , $U_{\alpha\beta}$, or $U_{\alpha\beta\gamma}$ by restricting F to the object, 1-morphism, or 2-morphism x_{α} , $x^{\alpha\beta}$, and $x^{\alpha\beta\gamma}$ respectively. To get the cocycle condition at a point x in $U_{\alpha\beta\gamma\delta}$ consider how F transforms the top diagram of 2-morphisms in $2X_{\mathcal{U}}$ into the bottom diagram of 2-morphisms in $2\mathcal{C}$:

$$\begin{array}{ccccc}
 & & x^{\gamma\delta} * (x^{\beta\gamma} * x^{\alpha\beta}) & \xrightarrow{\alpha} & (x^{\gamma\delta} * x^{\beta\gamma}) * x^{\alpha\beta} \\
 & \nearrow 1 * x^{\alpha\beta\gamma} & \downarrow x^{\alpha\beta\gamma\delta} & & \downarrow x^{\alpha\beta\delta} \\
 x^{\gamma\delta} * x^{\alpha\gamma} & & & & x^{\beta\delta} * x^{\alpha\beta} \\
 & \nwarrow x^{\alpha\gamma\delta} & & & \nwarrow x^{\beta\gamma\delta} * 1 \\
 & & x^{\alpha\delta} & & \\
 & & \downarrow F & & \downarrow F \\
 & & E^{\gamma\delta} * (E^{\beta\gamma} * E^{\alpha\beta}) & \xrightarrow{\alpha} & (E^{\gamma\delta} * E^{\beta\gamma}) * E^{\alpha\beta} \\
 & \nearrow 1 * \phi^{\alpha\beta\gamma} & \downarrow \phi^{\alpha\beta\gamma\delta} & & \downarrow \phi^{\alpha\beta\delta} \\
 E^{\gamma\delta} * E^{\alpha\gamma} & & & & E^{\beta\delta} * E^{\alpha\beta} \\
 & \nwarrow \phi^{\alpha\gamma\delta} & & & \nwarrow \phi^{\beta\gamma\delta} * 1 \\
 & & E^{\alpha\delta} & &
 \end{array}$$

From a charted $2\mathcal{C}$ -bundle one can construct F by first defining it on objects, 1-morphisms, and 2-morphisms of the form x_{α} , $x^{\alpha\beta}$, and $x^{\alpha\beta\gamma}$. We know that $2X_{\mathcal{U}}$ is generated by these 1- and 2-morphisms, and the associators. Thus the strict functor F is uniquely determined by the data above.

A map $U_\bullet \rightarrow \Delta 2\mathcal{C}$ is uniquely determined by the maps of k -simplices for $k \leq 3$. For the correspondence to charted $2\mathcal{C}$ -bundles the map on 0-simplices, 1-simplices, and 2-simplices give the V^α 's, the $E^{\alpha\beta}$'s, and the $\phi^{\alpha\beta\gamma}$'s respectively. The coherence condition on tetrahedrons in $\Delta 2\mathcal{C}$ corresponds to the cocycle condition of the charted $2\mathcal{C}$ -bundle. \square

Lemma 2.7 *Let $(\mathcal{U}, \mathcal{I})$ be an ordered open cover of X , then U_\bullet sits as a simplicial deformation retract of $\Delta 2X_\mathcal{U}$.*

One could probably prove this by identifying U_\bullet as the nerve of a category biequivalent to $2X_\mathcal{U}$ as 2-categories, see [Str96, section 10]. However, a direct argument is not difficult.

Proof: Observe that $\Delta 2X_\mathcal{U}$ actually is 2-cosimplicial, and we can write any n -simplex as a flag

$$\begin{pmatrix} x_{\alpha_0} & f_{01} & f_{02} & f_{03} & \cdots & f_{0n} \\ & x_{\alpha_1} & f_{12} & f_{13} & \cdots & f_{1n} \\ & & x_{\alpha_2} & f_{23} & \cdots & f_{2n} \\ & & & x_{\alpha_3} & \cdots & f_{3n} \\ & & & & \ddots & \vdots \\ & & & & & x_{\alpha_n} \end{pmatrix},$$

where $\alpha_0 < \alpha_1 < \dots < \alpha_n$, $x \in U_{\alpha_0 \dots \alpha_n}$ and $f_{ij} : x_{\alpha_i} \rightarrow x_{\alpha_j}$ are 1-morphisms in $2X_\mathcal{U}$ such that for every $i < j < k$ there exists a 2-morphism $f_{ik} \Rightarrow f_{jk} * f_{ij}$. Define $U_\bullet \rightarrow \Delta 2X_\mathcal{U}$ by sending the n -simplex $x \in U_{\alpha_0 \alpha_1 \dots \alpha_n}$ to the n -simplex in $\Delta 2X_\mathcal{U}$ given by $f_{ij} = x^{\alpha_i \alpha_j}$. Observe that this gives an inclusion, and for the rest of the proof we view U_\bullet as a sub-simplicial space of $\Delta 2X_\mathcal{U}$.

Now we define a simplicial deformation retraction h from $\Delta 2X_\mathcal{U}$ down to U_\bullet by

$$h_k(f_{ij}) = \begin{pmatrix} x_{\alpha_0} & \cdots & x^{\alpha_0 \alpha_{k-1}} & x^{\alpha_0 \alpha_k} & x^{\alpha_0 \alpha_k} & x^{\alpha_0 \alpha_{k+1}} & \cdots & x^{\alpha_0 \alpha_n} \\ & \ddots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ & & x_{\alpha_{k-1}} & x^{\alpha_{k-1} \alpha_k} & x^{\alpha_{k-1} \alpha_k} & x^{\alpha_{k-1} \alpha_{k+1}} & \cdots & x^{\alpha_{k-1} \alpha_n} \\ & & & x_{\alpha_k} & 1_{x_{\alpha_k}} & x^{\alpha_k \alpha_{k+1}} & \cdots & x^{\alpha_k \alpha_n} \\ & & & & x_{\alpha_k} & f_{kk+1} & \cdots & f_{kn} \\ & & & & & x_{\alpha_{k+1}} & \cdots & f_{k+1 n} \\ & & & & & & \ddots & \vdots \\ & & & & & & & x_{\alpha_n} \end{pmatrix}$$

These maps $h_k : (\Delta 2X_\mathcal{U})_n \rightarrow (\Delta 2X_\mathcal{U})_{n+1}$, for $0 \leq k \leq n$, verify the usual identities for a simplicial deformation retraction. \square

Given a charted $2\mathcal{C}$ -bundle \mathcal{E} over X and a subspace A of X it is clear that we may restrict \mathcal{E} to A . We denote the restriction by $\mathcal{E}|_A$. More generally, there is a pullback $f^*\mathcal{E}$ over Y , for every continuous map $f : Y \rightarrow X$.

We define equivalence of charted $2\mathcal{C}$ -bundles by concordance:

Definition 2.8 *Two charted $2\mathcal{C}$ -bundles \mathcal{E}_0 and \mathcal{E}_1 over X are equivalent (or concordant) if there exists a charted $2\mathcal{C}$ -bundle \mathcal{E} over $X \times I$ such that $\mathcal{E}|_{X \times \{i\}} = \mathcal{E}_i$, $i = 0, 1$.*

3 Classifying spaces

M. Weiss, [Wei05], has asked the question: What does the classifying space of a category classify? Here the classifying space of a connected discrete category, \mathcal{C} , is defined as the geometric realization of the nerve. However, only when \mathcal{C} is a groupoid do we know that $B\mathcal{C}$ classifies principal G -bundles, where G is the automorphism group of an object in \mathcal{C} .

When working with a 2-category $2\mathcal{C}$ we have on one hand the geometric nerve $\Delta 2\mathcal{C}$ and its geometric realization, while on the other hand we have charted $2\mathcal{C}$ -bundles. Hence, it is natural to ask: When does the geometric nerve of a topological 2-category $2\mathcal{C}$ classify charted $2\mathcal{C}$ -bundles? In this section we will describe sufficient conditions.

We will assume that our base space spaces have the homotopy type of a CW-complex.

Unlike what is customary amongst some author, we will not call anything a classifying space unless we know what it classifies. Therefore we define:

Definition 3.1 *If there exists a space B such that for all X having the homotopy type of a CW-complex, there is a natural one-to-one correspondence between concordance classes of charted $2\mathcal{C}$ -bundles over X and homotopy classes of maps $X \rightarrow B$, then we say that B is a classifying space for charted $2\mathcal{C}$ -bundles. We denote such classifying spaces by $B2\mathcal{C}$.*

Remark 3.2 *The set of charted $2\mathcal{C}$ -bundles over X can be made into a sheaf by placing the indexing set \mathcal{I} in a poset universe. Using techniques of I. Madsen and M. Weiss [MW04, proposition 2.4.3] there exists a classifying space as defined above.*

In order to prove our main theorem, we need conditions on $2\mathcal{C}$. We define:

Definition 3.3 *We call $2\mathcal{C}$ good if its geometric nerve $\Delta 2\mathcal{C}$ is a good simplicial space. We say that $2\mathcal{C}$ is sufficiently fibrant if its geometric nerve $\Delta 2\mathcal{C}$ satisfies the topological Kan condition, (see definition 5.1 below).*

In section 6 we will provide explicit criteria to ensure that $2\mathcal{C}$ is good, theorem 6.4, and sufficiently fibrant, theorem 6.7.

The following is our main theorem. Its proof is long and technical, so that will be postponed to the last four sections.

Theorem 3.4 *The geometric nerve of a good topological 2-category $2\mathcal{C}$ is a classifying space for charted $2\mathcal{C}$ -bundles.*

The correspondence sends a charted $2\mathcal{C}$ -bundle over X , given as a simplicial map $U_\bullet \rightarrow \Delta 2\mathcal{C}$, to the homotopy class of maps $X \rightarrow |\Delta 2\mathcal{C}|$ specified by Dugger and Isaksen's weak equivalence $X \xleftarrow{\simeq} |U_\bullet|$, see [DI04], and the induced map under geometric realization, $|U_\bullet| \rightarrow |\Delta 2\mathcal{C}|$.

Similarly, if we represent a charted $2\mathcal{C}$ -bundle by a strict 2-functor $F : 2X_{\mathcal{U}} \rightarrow 2\mathcal{C}$, then applying the geometric nerve and taking the geometric realization yields a map

$$|\Delta 2X_{\mathcal{U}}| \rightarrow |\Delta 2\mathcal{C}| \quad ,$$

where the source is naturally homotopy equivalent to X .

Let us consider a topological group G as the 1-morphisms of a topological 2-category with a single object and only trivial 2-morphisms. G is called *well-pointed* if the inclusion of its unit is a closed cofibration. In this case the geometric nerve, ΔG coincides with the ordinary nerve $N_\bullet G$, and well-pointed implies that the nerve of G is good. Hence we get the following well known result as a corollary:

Corollary 3.5 *If G is well-pointed, then principal G -bundles are classified by $BG = |N_\bullet G| = |\Delta G|$.*

Notice that our proof does not mention the universal bundle over the classifying space.

In the case where $2\mathcal{C}$ is discrete we get the following corollary:

Corollary 3.6 *The geometric nerve of a discrete 2-category $2\mathcal{C}$ is a classifying space for charted $2\mathcal{C}$ -bundles.*

In this case it is possible to give a more elementary proof using the following facts: We may assume that X is a CW-complex. Every ordered open cover of X has a good refinement. Let \mathcal{U} be a good ordered open cover. Collapsing each connected component of the ordered Čech complex U_\bullet to a point yields a simplicial set Y_\bullet and there are natural homotopy equivalences $X \xleftarrow{\simeq} |U_\bullet| \xrightarrow{\simeq} |Y_\bullet|$, see [DI04, theorem 2.1, proposition 2.6 and corollary A.6]. The geometric nerve $\Delta 2\mathcal{C}$ of a 2-groupoid is a Kan complex, see [Dus02, theorem 8.6].

When $2\mathcal{C}$ is a strict and discrete 2-category, the hom-sets are categories $2\mathcal{C}(x, y)$. Take nerves of these to get a simplicial category $N_\bullet 2\mathcal{C}$. Taking

the nerve again we get a bisimplicial set $N_{\bullet}N_{\bullet}2\mathcal{C}$. Bullejos and Cegarra has shown that for any strict and discrete 2-category there is a natural homotopy equivalence

$$|N_{\bullet}N_{\bullet}2\mathcal{C}| \simeq |\Delta 2\mathcal{C}| \quad ,$$

see [BC03, theorem 1.1]. Moreover, they have a generalization of Quillen's Theorem A to this setting:

Theorem 3.7 ([BC03, Theorem 1.2]) *Let $F : 2\mathcal{C} \rightarrow 2\mathcal{C}'$ be a 2-functor. If the double nerve $N_{\bullet}N_{\bullet}(y' // F)$ is contractible for every object $y' \in 2\mathcal{C}'$, then the induced map $N_{\bullet}N_{\bullet}2\mathcal{C} \rightarrow N_{\bullet}N_{\bullet}2\mathcal{C}'$ is a homotopy equivalence.*

Here $y' // F$ is Gray's homotopy fiber 2-category, or lax comma category, see [Gra80, section 3.1].

Remark 3.8 *When calculating $[X, |\Delta 2\mathcal{C}|]$, we can replace the topological 2-category $2\mathcal{C}$ with a simplicial 2-category. The standard way of doing this is to apply the functor Sing_{\bullet} to objects, 1- and 2-morphisms. Recall that for a topological space Y the simplicial set $\text{Sing}_{\bullet}(Y)$ has q -simplices the set of maps $\Delta^q \rightarrow Y$. Together with geometric realization this functor is an adjunction between topological spaces and simplicial sets. Moreover, the derived functors induce an equivalence between the associated homotopy categories, see for example [GJ99, theorem I.11.4]. In particular, the natural map $Y \rightarrow |\text{Sing}_{\bullet} Y|$ is a weak equivalence. Since we are assuming that X has the homotopy type of a CW-complex and that $2\mathcal{C}$ is good, it follows that there is a natural bijection*

$$[X, |\Delta 2\mathcal{C}|] \cong [X, |\Delta \text{Sing}_{\bullet} 2\mathcal{C}|] \quad .$$

4 Calculations

In this section we will investigate possible definitions of 2-vector bundles based on J. C. Baez and A. S. Crans' 2-vector spaces, see [BC04, section 3]. The topological K -theory of these 2-vector bundles splits as two copies of the K -theory of ordinary vector bundles. The main reason for this behavior is that Baez and Crans' 2-category of 2-vector spaces, which can be defined as 2-term chain complexes, admits a 2-retraction into the direct sum of two general linear groups. This is given by taking the homology of the chain complex. Moreover, the 2-morphisms in Baez and Crans' 2-vector spaces, i.e. the chain homotopies, between fixed 1-morphisms form affine spaces.

Suppose that $2\mathcal{C}$ is a topological 2-category. The calculation we are going to do relies on a list of assumptions. These are:

- A1.** $2\mathcal{C}$ is a strict 2-category.
- A2.** $2\mathcal{C}$ is a good topological 2-groupoid, i.e. every 1-morphism is an equivalence and every 2-morphism is an isomorphism. Good means that $\Delta 2\mathcal{C}$ is a good simplicial space.

A3. $2\mathcal{C}$ is continuously connected, i.e. the source-target map from the total space of 1-morphisms in $2\mathcal{C}$ to the product of two copies of the space of objects of $2\mathcal{C}$,

$$1 \operatorname{Mor}(2\mathcal{C}) \xrightarrow{(s,t)} \operatorname{Obj} 2\mathcal{C} \times \operatorname{Obj} 2\mathcal{C} \quad ,$$

is a surjective Serre fibration.

A4. For every pair of 1-morphisms $f, g : x_0 \rightarrow x_1$ sharing source and target, the space of 2-morphisms $2 \operatorname{Mor}(2\mathcal{C})(f, g)$ is either contractible or empty.

A5. There exists a well-pointed topological group G together with continuous 2-functors $G \xrightarrow{i} 2\mathcal{C} \xrightarrow{H} G$ such that Hi is the identity on G . Here i can even be a lax 2-functor.

A6. H detects 2-morphisms, i.e. if $f, g : x_0 \rightarrow x_1$ are 1-morphisms in $2\mathcal{C}$ sharing source and target, then $H(f) = H(g)$ implies the existence of a 2-morphism $\phi : f \Rightarrow g$.

The aim is to calculate the geometric nerve of $2\mathcal{C}$. By remark 3.8, we lose nothing by studying the simplicial 2-category $\operatorname{Sing}_\bullet 2\mathcal{C}$ instead. Notice also that $\operatorname{Sing}_\bullet$ commutes with geometric nerve, double nerve and ordinary nerve constructions.

To our topological 2-category we assign a homotopy category, $\operatorname{Ho}(2\mathcal{C})$. Under assumption **A2**, this topological category has a simple description. Its space of objects equals the objects of $2\mathcal{C}$. The space of morphisms $\operatorname{Mor}(\operatorname{Ho}(2\mathcal{C}))$ is the quotient space of $1 \operatorname{Mor}(2\mathcal{C})$ where we identify two 1-morphisms f and g if there exists a 2-morphism $F : f \Rightarrow g$. Since every 2-morphism in $2\mathcal{C}$ is an isomorphism, this is an equivalence relation. Let $[f]$ in $\operatorname{Mor}(\operatorname{Ho}(2\mathcal{C}))$ denote the equivalence class corresponding to f in $1 \operatorname{Mor}(2\mathcal{C})$. Observe that $\operatorname{Ho}(2\mathcal{C})$ is a groupoid under assumption **A2**. The quotient map induces a continuous strict 2-functor

$$\gamma : 2\mathcal{C} \rightarrow \operatorname{Ho}(2\mathcal{C}) \quad .$$

Lemma 4.1 *Under the assumptions **A1**, **A2**, and **A4**, there is a natural chain of weak equivalences*

$$|\Delta 2\mathcal{C}| \simeq \cdots \simeq |N_\bullet \operatorname{Ho}(2\mathcal{C})| \quad .$$

Proof: We are in the strict case, so it is enough to consider the double nerve. Up to weak equivalence we lose nothing by considering simplicial instead of topological 2-categories. Thus we inspect the induced map

$$N_\bullet N_\bullet \operatorname{Sing}_\bullet 2\mathcal{C} \rightarrow N_\bullet N_\bullet \operatorname{Sing}_\bullet \operatorname{Ho}(2\mathcal{C}) \quad .$$

Since pointwise weak equivalences of bisimplicial sets induce a weak equivalence on the geometric realization, we can drop the outer nerve and show that

$$N_{\bullet} \text{Sing}_{\bullet} 2\mathcal{C}(x, y) \rightarrow N_{\bullet} \text{Sing}_{\bullet} \text{Ho}(2\mathcal{C})(x, y)$$

is a weak equivalence for every pair of objects x, y in $2\mathcal{C}$. This would imply the statement of the lemma. Here $2\mathcal{C}(x, y)$ is the topological category with objects 1-morphisms $x \rightarrow y$ and morphisms the corresponding 2-morphisms. Similarly $\text{Ho}(2\mathcal{C})(x, y)$ is a topological space, viewed as a topological category with only identity morphisms. Again, it is enough to show that we have a weak equivalence before taking the nerve. Thus we consider

$$\text{Sing}_{\bullet} 2 \text{Mor}(2\mathcal{C})(f, g) \rightarrow 2 \text{Mor}(\text{Ho}(2\mathcal{C}))([f], [g]) \quad ,$$

where f and g are 1-morphisms in $2\mathcal{C}$ sharing source and target. If there exists a 2-morphism $F : f \Rightarrow g$, then $2 \text{Mor}(2\mathcal{C})(f, g)$ is contractible by assumption **A4**, whereas $2 \text{Mor}(\text{Ho}(2\mathcal{C}))([f], [g])$ consists of a single element, namely $id_{[f]} = id_{[g]}$. Otherwise, both spaces are empty. Hence the map above is a weak equivalence. \square

Lemma 4.2 *Under the assumptions **A2**, **A3**, **A5** and **A6**, the 2-functor $H : 2\mathcal{C} \rightarrow G$ induces a weak equivalence*

$$|N_{\bullet} \text{Ho}(2\mathcal{C})| \xrightarrow{\cong} |N_{\bullet} G| \quad .$$

Proof: Again we may pass to the simplicial setting by applying Sing_{\bullet} . It is enough to show that we have weak equivalences

$$N_{\bullet} \text{Sing}_q \text{Ho}(2\mathcal{C}) \rightarrow N_{\bullet} \text{Sing}_q G$$

for fixed simplicial degree q . At the level of the homotopy category assumption **A5** induces functors

$$G \xrightarrow{i} \text{Ho}(2\mathcal{C}) \xrightarrow{H} G$$

with $Hi = id_G$. Taking singular q -simplices we get

$$\text{Sing}_q G \xrightarrow{i} \text{Sing}_q \text{Ho}(2\mathcal{C}) \xrightarrow{H} \text{Sing}_q G \quad .$$

Again $Hi = id$. We are done if we can find a natural transformation from iH to the identity on $\text{Sing}_q \text{Ho}(2\mathcal{C})$.

An object x of $\text{Sing}_q \text{Ho}(2\mathcal{C})$ is a map $x : \Delta^q \rightarrow \text{Obj } 2\mathcal{C}$. For every such x , assumption **A3** assures that we may choose a lift \tilde{f}_x in the diagram

$$\begin{array}{ccc} & & 1 \text{Mor}(2\mathcal{C}) \\ & \nearrow \tilde{f}_x & \downarrow (s,t) \\ \Delta^q & \xrightarrow{(iHx,x)} & \text{Obj } 2\mathcal{C} \times \text{Obj } 2\mathcal{C} \end{array} \quad .$$

The class $[\tilde{f}_x]$ is a morphism $iH(x) \rightarrow x$ in $\text{Sing}_q \text{Ho}(\mathcal{2C})$.

Let us now modify \tilde{f}_x into a map $f_x : iH(x) \rightarrow x$ with $H(f_x) : \Delta^q \rightarrow G$ constantly equal to $1 \in G$. This is simply done by setting

$$f_x = \tilde{f}_x i(H(\tilde{f}_x)^{-1}) \quad .$$

We now claim that the collection of all $[f_x]$ is a natural transformation from iH to the identity on $\text{Sing}_q \text{Ho}(\mathcal{2C})$. To see this we need to verify that the following diagram in $\text{Sing}_q \text{Ho}(\mathcal{2C})$ commutes for every morphism $[f] : x \rightarrow y$:

$$\begin{array}{ccc} iH(x) & \xrightarrow{iH([f])} & iH(y) \\ [f_x] \downarrow & & \downarrow [f_y] \\ x & \xrightarrow{[f]} & y \end{array} \quad .$$

Commutativity may be checked pointwise for $\mathbf{t} \in \Delta^q$, and at \mathbf{t} we consider the above diagram lifted to $\mathcal{2C}$:

$$\begin{array}{ccc} iH(x(\mathbf{t})) & \xrightarrow{iH(f(\mathbf{t}))} & iH(y(\mathbf{t})) \\ f_x(\mathbf{t}) \downarrow & & \downarrow f_y(\mathbf{t}) \\ x(\mathbf{t}) & \xrightarrow{f(\mathbf{t})} & y(\mathbf{t}) \end{array} \quad .$$

Applying H we get a commutative diagram in G :

$$\begin{array}{ccc} H(x(\mathbf{t})) & \xrightarrow{H(f(\mathbf{t}))} & H(y(\mathbf{t})) \\ 1=H(f_x(\mathbf{t})) \downarrow & & \downarrow H(f_y(\mathbf{t}))=1 \\ H(x(\mathbf{t})) & \xrightarrow{H(f(\mathbf{t}))} & H(y(\mathbf{t})) \end{array}$$

Since H detects 2-morphisms, assumption **A6**, the diagram in $\mathcal{2C}$ commutes up to a 2-morphism. Thus the first diagram commutes in the homotopy category. Hence, we have a natural transformation as claimed. This finishes the proof. \square

Thus we have proved:

Theorem 4.3 *Under the assumptions **A1**, **A2**, **A3**, **A4**, **A5** and **A6** above the classifying space of $\mathcal{2C}$ is weakly equivalent to BG .*

It is also possible, but less enlightening, to prove this directly, using Bullejos and Cegarra's version of Quillen's Theorem A, see theorem 3.7, and arguing using the homotopy fiber categories of $i : G \rightarrow \mathcal{2C}$.

Let us now discuss choices of $\mathcal{2C}$ based on Baez and Crans' 2-vector spaces. The numerous variations depend on how strictly the 1-morphisms

should be equivalences, how much structure the 2-morphisms should contain, and how to choose the objects from 2-vector spaces within the same weak equivalence class. Let n , b_0 and b_1 be positive integers and let C_* be a 2-term chain complex of vector spaces.

Let $2\mathcal{B}_{\text{strict}}(C_*)$ be the 2-category with a single object C_* , the 1-morphisms are the chain isomorphisms $f : C_* \rightarrow C_*$, and the 2-morphisms $f \rightarrow g$ are chain homotopies ϕ from f to g .

Let $2\mathcal{B}_{\text{weak}}(C_*)$ be the 2-category with a single object C_* , the 1-morphisms are the chain equivalences $f : C_* \rightarrow C_*$, and the 2-morphisms $f \rightarrow g$ are chain homotopies ϕ from f to g .

Let $2\mathcal{B}_{\text{eq}}(C_*)$ be the 2-category with a single object C_* , the 1-morphisms are tuples $(f, \bar{f}, \iota_f, \epsilon_f)$, where f and \bar{f} are chain maps $C_* \rightarrow C_*$, and $\iota_f : 1 \rightarrow f\bar{f}$ and $\epsilon_f : \bar{f}f \rightarrow 1$ are chain homotopies, and the 2-morphisms $(f, \bar{f}, \iota_f, \epsilon_f) \rightarrow (g, \bar{g}, \iota_g, \epsilon_g)$ are pairs of chain homotopies $\phi : f \rightarrow g$ and $\bar{\phi} : \bar{f} \rightarrow \bar{g}$ such that the following identities hold:

$$\iota_g - \iota_f = f\bar{\phi} + \phi\bar{g} \quad \text{and} \quad \epsilon_f - \epsilon_g = f\bar{\phi} + \phi\bar{g} \quad .$$

Let $2\mathcal{B}_{\text{ad}}(C_*)$ be the 2-category with a single object C_* , the 1-morphisms are tuples $(f, \bar{f}, \iota_f, \epsilon_f)$, where f and \bar{f} are chain maps $C_* \rightarrow C_*$, and $\iota_f : 1 \rightarrow f\bar{f}$ and $\epsilon_f : \bar{f}f \rightarrow 1$ are chain homotopies such that the zigzag identities

$$f\epsilon_f + \iota_f f = 0 \quad \text{and} \quad \epsilon_f \bar{f} + \bar{f}\epsilon_f = 0$$

hold, and the 2-morphisms $(f, \bar{f}, \iota_f, \epsilon_f) \rightarrow (g, \bar{g}, \iota_g, \epsilon_g)$ are pairs of chain homotopies $\phi : f \rightarrow g$ and $\bar{\phi} : \bar{f} \rightarrow \bar{g}$ such that the following identities hold:

$$\iota_g - \iota_f = f\bar{\phi} + \phi\bar{g} \quad \text{and} \quad \epsilon_f - \epsilon_g = f\bar{\phi} + \phi\bar{g} \quad .$$

Let $2\mathcal{B}_{\text{weak}}^H(b_0, b_1)$ be the 2-category with objects all 2-term chain complexes C_* with Betti-numbers b_0, b_1 , the 1-morphisms are the chain equivalences $f : C_* \rightarrow C'_*$, and the 2-morphisms $f \rightarrow g$ are chain homotopies ϕ from f to g .

Let $2\mathcal{B}_{\text{eq}}^H(b_0, b_1)$ be the 2-category with objects all 2-term chain complexes C_* with Betti-numbers b_0, b_1 , the 1-morphisms are tuples $(f, \bar{f}, \iota_f, \epsilon_f)$, where $f : C_* \rightarrow C'_*$ and $\bar{f} : C'_* \rightarrow C_*$ are chain maps, and $\iota_f : 1 \rightarrow f\bar{f}$ and $\epsilon_f : \bar{f}f \rightarrow 1$ are chain homotopies, and the 2-morphisms $(f, \bar{f}, \iota_f, \epsilon_f) \rightarrow (g, \bar{g}, \iota_g, \epsilon_g)$ are pairs of chain homotopies $\phi : f \rightarrow g$ and $\bar{\phi} : \bar{f} \rightarrow \bar{g}$ such that the following identities hold:

$$\iota_g - \iota_f = f\bar{\phi} + \phi\bar{g} \quad \text{and} \quad \epsilon_f - \epsilon_g = f\bar{\phi} + \phi\bar{g} \quad .$$

Let $2\mathcal{B}_{\text{ad}}^H(b_0, b_1)$ be the 2-category with objects all 2-term chain complexes C_* with Betti-numbers b_0, b_1 , the 1-morphisms are tuples $(f, \bar{f}, \iota_f, \epsilon_f)$, where $f : C_* \rightarrow C'_*$ and $\bar{f} : C'_* \rightarrow C_*$ are chain maps, and $\iota_f : 1 \rightarrow f\bar{f}$ and $\epsilon_f : \bar{f}f \rightarrow 1$ are chain homotopies such that the zigzag identities

$$f\epsilon_f + \iota_f f = 0 \quad \text{and} \quad \epsilon_f \bar{f} + \bar{f}\epsilon_f = 0$$

hold, and the 2-morphisms $(f, \bar{f}, \iota_f, \epsilon_f) \rightarrow (g, \bar{g}, \iota_g, \epsilon_g)$ are pairs of chain homotopies $\phi : f \rightarrow g$ and $\bar{\phi} : \bar{f} \rightarrow \bar{g}$ such that the following identities hold:

$$\iota_g - \iota_f = f\bar{\phi} + \phi\bar{g} \quad \text{and} \quad \epsilon_f - \epsilon_g = f\bar{\phi} + \phi\bar{g} \quad .$$

Let $2\mathcal{B}$ be one of the topological 2-categories above. Observe that in all cases there is an inclusion 2-functor

$$GL(b_1) \times GL(b_0) \rightarrow 2\mathcal{B} \quad .$$

There is a retraction to this inclusion given as the homology of the 2-term chain complex C_* . Furthermore, this retraction satisfies the assumptions. In particular, **A4** holds because $2\text{Mor}(2\mathcal{B})(f, g)$ are affine spaces, while **A6** relies on:

Lemma 4.4 ([Spa91, exercise 4.A.3]) *Let C_* and C'_* be chain complexes of complex vector spaces. If $f, g : C_* \rightarrow C'_*$ are chain maps which induce the same homomorphism of homology groups, then there exists a chain homotopy $\phi : f \simeq g$.*

Proof: Notice that the homology groups are vector spaces. Let B_q be the image of ∂_{q+1} in C_q . B_q is contained in the kernel of ∂_q , and we choose a splitting $\ker \partial_q \cong B_q \oplus H_q$. We identify H_q with the q 'th homology vector space of C_* . Furthermore, we may split C_q as $\ker \partial_q \oplus B_{q-1}$. Hence we have chosen for each q an isomorphism

$$C_q \cong B_q \oplus H_q \oplus B_{q-1} \quad .$$

With this splitting ∂_q maps the summand B_{q-1} of C_q isomorphically onto B_{q-1} of C_{q-1} , whereas ∂_q sends the other summands to 0. Since f and g induce the same map in homology, $f(z) - g(z)$ is a boundary for all $z \in \ker \partial_q$. Hence there exists a map $\tilde{\phi}_q : H_q \rightarrow C'_{q+1}$ such that $\partial'_{q+1}\tilde{\phi}_q = (f_q - g_q)|_{H_q}$. Assume that $c \in C_q$ corresponds to $(b, h, b') \in B_q \oplus H_q \oplus B_{q-1}$. Define the chain homotopy $\phi : f \simeq g$ by

$$\phi_q(c) = f_{q+1}b - g_{q+1}b + \tilde{\phi}_q h \quad .$$

□

After this detour, let us consider the case $2\mathcal{B}_{\text{weak}}(C_*)$ in more detail:

Example 4.5 *As in the lemma above, we write the 2-term chain complex C_* as*

$$0 \rightarrow H_1 \oplus B \xrightarrow{\begin{pmatrix} 0 & id_B \\ 0 & 0 \end{pmatrix}} B \oplus H_0 \rightarrow 0 \quad ,$$

where H_1 and H_0 are vector spaces of dimension b_1 and b_0 respectively. Furthermore, H_1 and H_0 are identified with the homology groups of C_ . This defines the 2-functor H , and lemma 4.4 gives us **A6**. The inclusion 2-functor i is given by sending a pair of invertible matrices (M_1, M_0) in $GL(b_1) \oplus GL(b_0)$ to the chain map f with $f_1 = \begin{pmatrix} M_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $f_0 = \begin{pmatrix} 0 & 0 \\ 0 & M_0 \end{pmatrix}$.*

In general we have:

Corollary 4.6 *Let X have the homotopy type of a CW-complex and $2\mathcal{B}$ be one of the topological 2-categories above. There is a bijection between concordance classes of $2\mathcal{B}$ -bundles over X and a pair of concordance classes of ordinary vector bundles over X of fiber dimension b_0 and b_1 respectively. Consequently the 2- K -theory associated to Baez and Crans' 2-vector spaces splits as two copies of ordinary K -theory.*

Thus we may write

$$2K_{\text{Baez}} \simeq K \times K \quad ,$$

where K is ordinary topological K -theory of vector bundles.

However, based on Kapranov and Voevodsky's 2-vector spaces we could define $2\mathcal{C}$ to be one of the following:

Let $2\mathcal{KV}_{\text{pt}}(n)$ be the topological 2-category of example 2.3.

Let $2\mathcal{KV}(n)$ be the topological 2-category with n -tuples $V = (V_1, V_2, \dots, V_n)$ of complex vector spaces as objects, the 1-morphisms $V \rightarrow V'$ are $n \times n$ -matrices $E = (E_{ij})$ with $\det \dim E = \pm 1$ and $E \cdot V = V'$, and the 2-morphisms $E \rightarrow E'$ are $n \times n$ -matrices $\phi = (\phi_{ij})$ of isomorphisms $\phi_{ij} : E_{ij} \xrightarrow{\cong} E'_{ij}$.

In this case the 2-category is not a 2-groupoid, so theorem 4.3 does not apply. Furthermore, Baas, Dundas and Rognes has calculated the associated 2- K -theory in [BDR04] and shown that this generalized cohomology theory is $K_{\text{alg}}(ku)$, and essentially of chromatic filtration 2, thus resembling something elliptic. We have:

$$2K_{\text{BDR}} \simeq K_{\text{alg}}(ku)$$

If $\mathcal{V} = \text{FinSet}$ (the category of finite sets and maps) and $2\mathcal{C} = \mathcal{V} - \text{mod}$, then the representing spectrum is $K(\mathbb{S}) = A(*)$, relating this to Waldhausen's A -theory (as pointed out by John Rognes).

5 The topological Kan condition

We consider a simplicial space Z_\bullet with a topological Kan property. To be precise, in the notation of J. W. Duskin, we define the *space of k -horns in dimension n* to be

$$\begin{aligned} \Lambda_n^k(Z_\bullet) = & \\ \{(z_0, \dots, z_{k-1}, -, z_{k+1}, \dots, z_n) \mid d_i(z_j) = d_{j-1}(z_i), i < j, i \neq k \neq j\} & \\ \subset \underbrace{Z_{n-1} \times Z_{n-1} \times \dots \times Z_{n-1}}_{n \text{ factors}} & \end{aligned}$$

For our purposes, it seems more natural to associate this horn to the complementary set $I(k) = \{0, 1, \dots, k-1, k+1, \dots, n\}$. So we think of a k -horn as an $I(k)$ -cohorn. If $I \subset [n] = \{0, 1, \dots, n\}$, we define the *space of I -cohorns* to be

$$\begin{aligned} C\Lambda_n^I(Z_\bullet) = & \\ \{(z_i)_{i \in I} \mid d_i(z_j) = d_{j-1}(z_i), i < j, i \in I \ni j\} & \\ \subset \underbrace{Z_{n-1} \times Z_{n-1} \times \dots \times Z_{n-1}}_{|I| \text{ factors}} & \end{aligned}$$

Note that $\Lambda_n^k = C\Lambda_n^{I(k)}$. There is a canonical map $c_n^{I(k)} : Z_n \rightarrow \Lambda_n^k(Z_\bullet) = C\Lambda_n^{I(k)}(Z_\bullet)$, and more generally, for any $I \subset [n]$ we have a canonical map $c_n^I : Z_n \rightarrow C\Lambda_n^I(Z_\bullet)$.

Definition 5.1 *We say that Z_\bullet satisfies the discrete Kan condition in dimension n if c_k is surjective for all $0 \leq k \leq n$. We say that Z_\bullet satisfies the topological Kan condition in dimension n if the maps c_k are surjective Serre fibrations for all $0 \leq k \leq n$.*

Lemma 5.2 *If the simplicial space Z_\bullet satisfies the topological Kan condition both in dimension n and in $n-1$, then for every non-empty $I \subset [n]$ the map c_n^I is a surjective Serre fibration.*

Proof: If $|I| = n-1$, then $I = I(k)$ for some $0 \leq k \leq n$, and the lemma follows directly from the definitions. Suppose inductively that the lemma is true for all $J \subset [n]$ with $n - |J| < n - |I| = m > 1$. Assume that $k \notin I$ and $J = I \cup \{k\}$, so that by the induction hypothesis c_n^J is a surjective Serre fibration. Consider the canonical map which forgets z_k

$$r_J^I : C\Lambda_n^J(Z_\bullet) \rightarrow C\Lambda_n^I(Z_\bullet).$$

Then $c_n^I = c_n^J \circ r_J^I$. So by the induction assumption, it is enough to show that the map r_J^I is a surjective Serre fibration.

Recall that $\delta_k: [n-1] \rightarrow [n]$ is the injective map which omits k , so that the face map $d_k: Z_n \rightarrow Z_{n-1}$ is the map induced by δ_k . Let the sequence $z = (z_i)_{i \in I}$ be a point in $C\Lambda^I(Z_\bullet)$. We can apply the face map d_k to each entry. We obtain a sequence of points in Z_{n-1} , namely $(d_k(z_i))_{i \in I}$. Let $I' \subset [n-1]$ be the subset determined by that $I = \partial_k(I')$. There is such a subset, since $k \notin I$. Moreover

$$|I'| = |I|$$

and the map

$$\delta_k: I' \rightarrow I$$

is a bijection. We define elements

$$y_i = \begin{cases} d_{k-1}(z_{\delta_k(i)}) = d_{k-1}(z_i) & \text{if } i \in I', \text{ and } i < k. \\ d_k(z_{\delta_k(i)}) = d_k(z_{i+1}) & \text{if } i \in I', \text{ and } k \leq i. \end{cases}$$

We claim that the sequence $(y_i)_{i \in I'}$ forms an I' -cohorn in Z_{n-1} , which we can fill by induction. This can be seen either by meditating on the geometry of the simplex, or more computationally in the following way (for $i, j \in I'$ and $i < j$):

$$d_i y_j = \begin{cases} d_i d_{k-1}(z_j) = d_{k-2} d_i(z_j) & i < j < k \\ d_i d_k(z_{j+1}) = d_{k-1} d_i(z_{j+1}) & i < k \leq j \\ d_i d_k(z_{j+1}) = d_k d_{i+1}(z_{j+1}) & k \leq i < j \end{cases}$$

and also (using that z is in the k^{th} I -cohorn):

$$d_{j-1} y_i = \begin{cases} d_{j-1} d_{k-1}(z_i) = d_{k-2} d_{j-1}(z_i) = d_{k-2} d_i z_j & i < j \leq k \\ d_{j-1} d_{k-1}(z_i) = d_{k-1} d_j(z_i) = d_{k-1} d_i z_{j+1} & i < k < j \\ d_{j-1} d_k(z_{i+1}) = d_k d_j(z_{i+1}) = d_{i+1} z_{j+1} & k \leq i < j \end{cases}$$

The association

$$z = (z_i)_{i \in I} \mapsto P^{I,k}(z) = (y_i)_{i \in I'}$$

defines a continuous map of horns $P^{I,k}: C\Lambda_n^I(Z_\bullet) \rightarrow C\Lambda_{n-1}^{I'}(Z_\bullet)$. Since $k \in J$ there is also a continuous map $p^k: C\Lambda_n^J(Z_\bullet) \rightarrow Z_n$ defined by picking out the appropriate component: $p^k((z_i)_{i \in J}) = z_k$.

These maps fit into a diagram of spaces

$$\begin{array}{ccc} C\Lambda_n^J(Z_\bullet) & \xrightarrow{r_J^I} & C\Lambda_n^I(Z_\bullet) \\ \downarrow p^k & & \downarrow P^{I,k} \\ Z_n & \xrightarrow{c_{n-1}^{I'}} & C\Lambda_{n-1}^{I'}(Z_\bullet) \end{array}$$

This is actually a pullback diagram. The bottom map is a surjective Serre fibration by assumption, so it follows that the top map is also a surjective Serre fibration. \square

We can now prove our main theorem.

Theorem 5.3 *Let Z_\bullet be a simplicial space.*

- *Assume that Z_\bullet satisfies the topological Kan condition in dimensions n and $n - 1$. Then the diagonal simplicial space $T_\bullet = \delta_\bullet(\text{Sing}_\bullet(Z))$ satisfies the Kan condition in dimension n .*
- *Assume that the three boundary maps $d_i: Z_2 \rightarrow Z_1$ are Serre fibrations, and assume that Z_\bullet satisfies the discrete Kan extension condition in dimension 2. Then the diagonal simplicial space $T_\bullet = \delta_\bullet(\text{Sing}_\bullet(Z))$ satisfies the Kan condition in dimension 2.*

Proof: We need to show that any horn in T_\bullet can be filled. The Kan condition is always satisfied for $n = 1$, since a 1-horn is filled by the degeneracy map $s_0: Z_0 \rightarrow Z_1$. So we assume without restriction that $n \geq 2$. A k -horn in dimension n in T_\bullet consists of a sequence of continuous maps

$$z_i: \Delta_{n-1} \rightarrow Z_{n-1} \quad \text{defined for } 0 \leq i \leq n-1, i \neq k,$$

with the property that if z_i and z_j are defined and $i < j$ then the following two continuous maps agree:

$$\begin{aligned} \Delta_{n-2} &\xrightarrow{\delta_i} \Delta_{n-1} \xrightarrow{z_j} Z_{n-1} \xrightarrow{d_i} Z_{n-2} \\ \Delta_{n-2} &\xrightarrow{\delta_{j-1}} \Delta_{n-1} \xrightarrow{z_i} Z_{n-1} \xrightarrow{d_{j-1}} Z_{n-2} \end{aligned}$$

Here, and in the rest of the proof, we abuse notation by not distinguishing between a map $[m] \rightarrow [n]$ and its affine extension to a continuous map $\Delta_m \rightarrow \Delta_n$. The way to visualize this, is to consider the topological horn

$$\Lambda_n^k = \cup_{i \neq k} \delta_i(\Delta_{n-1}) \subset \Delta_n$$

The map z_i can be identified with a map from the subset $\delta_i(\Delta_{n-1})$ of Λ_n^k . The maps corresponding to z_i and z_j do not necessarily agree on the intersection $\delta_i(\Delta_{n-1}) \cap \delta_j(\Delta_{n-1})$ of these subsets, but satisfy a more complicated compatibility condition.

Main step. We define a continuous map

$$y: \Lambda_n^k \rightarrow Z_n$$

such that for every $i \neq k$ we have that $z_i = d_i \circ y \circ \delta_i$. We define this map inductively. The induction is done over a sequence of subspaces of the skeletons of Λ_n^k , which we define now. Let W_r be the union of all r -dimensional subsimplices of Δ_n which contain the vertex k . Then

$$\{k\} = W_0 \subset W_1 \subset \dots \subset W_{n-1} = \Lambda_n^k \quad .$$

Start of induction: W_0 is included in every subspace $\partial_i(\Delta_n)$, so we obtain a sequence of elements $a_i = z_i \partial_i(k) \in Z_n$. These elements satisfy that if $i < j$, $i \neq k \neq j$, then

$$d_i a_j = d_i z_j(\partial_j(k)) = d_{j-1} z_i(\partial_j(k)) = d_{j-1} a_i \quad .$$

By assumption the map c_k is surjective, so we can find an element $b \in Z_{n+1}$ such that $d_i(b) = a_i$. We define y on $W_0 = \{k\}$ by $y(k) = b \in Z_{n+1}$.

Induction step: Assume that we have defined $y: W_{r-1} \rightarrow Z_{n+1}$ such that for each $i \neq k$ we have a commutative diagram:

$$\begin{array}{ccccc} W_{r-1} \cap \partial_i(\Delta_{n-1}) & \hookrightarrow & W_{r-1} & \xrightarrow{y} & Z_n \\ \downarrow & & & & \downarrow d_i \\ \Delta_{n-1} & \xrightarrow{z_i} & & & Z_{n-1} \end{array}$$

We want to extend y to a map from W_r with the same property. To define this extension, it is sufficient to define it on each r -simplex in W_r . Such a simplex is determined by a subset $I \subset [n]$ with the two properties $k \in I$ and $|I| = r$. There is a unique order preserving injection with image I . $\alpha: [r] \rightarrow I \subset [n+1]$, and some k' , $0 \leq k' \leq r$ so that $\alpha(k') = k$. Using α , we can translate the problem of extending the map y to the simplex determined by I into a problem of constructing a map $y': \Delta_r \rightarrow Z_n$ with certain properties. In order to construct the extension of y to W_r , we have to solve this transformed problem for all injective $\alpha: [r] \rightarrow [n]$ such that the image $I = \alpha([r])$ contains k . We have to be careful about what the transformed problem is.

First, note that $\alpha(\Delta_r) \cap W_{r-1} = \alpha(\Lambda_r^{k'})$, so y' is already defined on $\Lambda_r^{k'}$.

There are two cases for the maps z_i . If $i \notin I$, then z_i is defined on a subspace $\alpha(\Lambda_r^k)$, and by induction, there it agrees with $d_i y$. So these maps do not give extra conditions on y' , beyond that it has to be an extension of the already defined map $y': \Lambda_r^{k'}$. On the other hand, if $i \in I$, say $i = \alpha(i')$, we do get an extra condition. First note that we have a canonical factorization

$$\begin{array}{ccc} [r-1] & \xrightarrow{\alpha_{i'}} & [n-1] \\ \downarrow \delta_{i'} & & \downarrow \delta_i \\ [r] & \xrightarrow{\alpha} & [n] \end{array}$$

where

$$\alpha'_{i'}(j') = \begin{cases} \alpha(j') & \text{if } j' < i', \\ \alpha(j' + 1) - 1 & \text{if } j' \geq i'. \end{cases}$$

For $i = \alpha(i')$ and $i \neq k$ we define

$$z_{\alpha,i}: \Delta_{r-1} \xrightarrow{\alpha'_{i'}} \Delta_{n-1} \xrightarrow{z_i} Z_{n-1}$$

We claim that these maps fit together to a map $z_\alpha: \Delta_r \rightarrow C\Lambda_n^I(Z_\bullet)$. To see this, assume that $0 \leq i' < j' \leq r$, $i = \alpha(i')$, $j = \alpha(j')$ and $i \neq k \neq j$. There is a commutative diagram (whose diagonal composite is $\alpha: [r] \rightarrow [n]$).

$$\begin{array}{ccccc} [r] & & & & \\ & \searrow^{\alpha'_j} & & & \\ & & [n-2] & \xrightarrow{\delta_i} & [n-1] \\ & \searrow^{\alpha''_{i,j}} & & & \downarrow \delta_j \\ & \searrow^{\alpha'_i} & & & \\ & & [n-1] & \xrightarrow{\delta_i} & [n] \end{array}$$

Taking this diagram into account, we compute:

$$\begin{aligned} d_i z_{\alpha,j} &= d_i z_j \alpha'_{j'} \\ &= d_{j-1} z_i \delta_{j-1} \alpha''_{i,j} \end{aligned}$$

and

$$\begin{aligned} d_{j-1} z_{\alpha,i} &= d_{j-1} z_i \alpha'_{i'} \\ &= d_{j-1} z_i \delta_{j-1} \alpha''_{i,j} \end{aligned}$$

End of argument for $n \geq 3$.

So the maps $z_{\alpha,i}$ collectively define a map

$$z_\alpha: \Delta^r \rightarrow C\Lambda_n^I(Z_\bullet)$$

The extension problem we have to solve is the following:

$$\begin{array}{ccc} \Lambda_k^s & \xrightarrow{y} & Z_{n+1} \\ \downarrow & \nearrow & \downarrow c_I \\ \Delta_r & \xrightarrow{z_\alpha} & C\Lambda_n^I Z_\bullet \end{array}$$

However, this is solvable, since the map c_I is a Serre fibration by lemma 5.2.

The case $n = 2$. Let $0 \leq k \leq 2$, and consider a k -horn in $\delta_1(\text{Sing}_\bullet(Z_\bullet))$. Use the Kan extension property at the vertex k to obtain a point in Z_2 . Then use that $d_i: Z_2 \rightarrow Z_1$ is a Serre fibration to obtain a continuous map $y: \Lambda_2^k \rightarrow Z_2$. Finally extend this to the 2-simplex.

In this case, we do not need lemma 5.2. \square

6 Good and sufficiently fibrant 2-groupoids

In this section we will give criteria on a topological 2-category $2\mathcal{C}$ to ensure that its geometric nerve $\Delta 2\mathcal{C}$ is good, and in the case $2\mathcal{C}$ is a topological 2-groupoid, we will give criteria that ensure that $\Delta 2\mathcal{C}$ is sufficiently fibrant.

We will use the notion of a *locally equiconnected* (abbr. LEC) space X , i.e. the diagonal map $X \rightarrow X \times X$ is a closed cofibration. All CW-complexes are LEC spaces by E. Dyer and S. Eilenberg's adjunction theorem, see [DE72]. Also by their results, we have:

Theorem 6.1 (Dyer, Eilenberg) *Assume that X is a LEC space, and $A \subseteq X$ admits a retraction $r: X \rightarrow A$. Then $i: A \hookrightarrow X$ is a closed cofibration, and A is a LEC space.*

Proof: By [DE72, theorem II.7] it is enough to show that A is a halo retract of X , i.e. A is a retract and there exists a continuous function $\psi: X \rightarrow I$ with $A = \psi^{-1}(0)$. We claim that every retract of a LEC space has a halo. From LEC data for X , we get a function $k: X \times X \rightarrow I$ with $k^{-1}(0)$ equal to the diagonal. Now define ψ by

$$\psi(x) = k(x, ir(x)) \quad .$$

It is easy to see that this is a halo of A . \square

Corollary 6.2 *A simplicial space Z_\bullet where all Z_q are LEC spaces is a good simplicial space.*

Proof: Any degeneracy $s_i: Z_q \rightarrow Z_{q+1}$ has a retract, namely d_i . Hence, s_i is a closed cofibration. \square

Our basic tool to prove that a space is LEC is:

Theorem 6.3 ([Hea86]) *Assume that X , B and E are LEC spaces. Then the pullback of $X \xrightarrow{f} B \xleftarrow{p} E$ is a LEC space provided that p is a Hurewicz fibration.*

Let us now list the criteria on a topological 2-category $2\mathcal{C}$ that imply that $2\mathcal{C}$ is good:

- G1.** The map $(-)^{-1} : 2\text{Mor} \rightarrow 2\text{Mor}$ sending a 2-morphism to its inverse exists and is continuous.
- G2.** The total space of 2-morphisms, 2Mor , is LEC.
- G3.** The source map $s_1 : 1\text{Mor} \rightarrow \text{Obj}$ from 1-morphisms to objects is a Hurewicz fibration.
- G4.** The target map $t_2 : 2\text{Mor} \rightarrow 1\text{Mor}$ from 2-morphisms to 1-morphisms is a Hurewicz fibration.

Theorem 6.4 *Let $2\mathcal{C}$ be a topological 2-category. If $2\mathcal{C}$ satisfies the assumptions **G1**, **G2**, **G3**, and **G4**, then the geometric nerve $\Delta 2\mathcal{C}$ is a good simplicial space.*

Before giving the proof, we will characterize n -simplices when $2\mathcal{C}$ satisfies **G1**. We have the following:

Lemma 6.5 *Let $2\mathcal{C}$ be a topological 2-category. Assume that the map taking a 2-morphism to its inverse exists and is continuous, then the space of n -simplexes $(\Delta 2\mathcal{C})_n$ is homeomorphic to the space of flags*

$$\begin{pmatrix} x_0 & f_{01} & \phi^{012} & \phi^{013} & \phi^{014} & \phi^{015} & \dots & \phi^{01n} \\ & x_1 & f_{12} & \phi^{123} & \phi^{124} & \phi^{125} & \dots & \phi^{12n} \\ & & x_2 & f_{23} & \phi^{234} & \phi^{235} & \dots & \phi^{23n} \\ & & & x_3 & f_{34} & \phi^{345} & \dots & \phi^{34n} \\ & & & & x_4 & f_{45} & \dots & \phi^{45n} \\ & & & & & x_5 & \dots & \phi^{56n} \\ & & & & & & \ddots & \vdots \\ & & & & & & & x_n \end{pmatrix}$$

where x_0, x_1, \dots, x_n are objects of $2\mathcal{C}$, the $f_{i+1} : x_i \rightarrow x_{i+1}$ are 1-morphisms, and $\phi^{i+1j} : f_{ij} \Rightarrow f_{i+1j} * f_{i+1}$, $i+1 < j$, are 2-morphisms.

Proof: The 1-morphisms f_{ij} with $i+1 < j$ are implicitly defined as the source of the 2-morphism ϕ^{i+1j} .

Given a flag of the form above, the only data of an n -simplex in $\Delta 2\mathcal{C}$ missing are the ϕ^{ijk} with $0 \leq i < j < k \leq n$ and $j \neq i+1$. We claim that these may be reconstructed continuously from the flag and the coherence condition of the geometric nerve.

Define the *valence* of the symbol ϕ^{ijk} , $0 < i < j < k < n$, to be the difference $j - i$. Observe that the flag contains all ϕ^{ijk} of valence 1. A

subtetrahedron, of the n -simplex under consideration, is determined by indices $0 \leq i < j < k < l \leq n$. Rewrite the coherence condition for such a subtetrahedron as the equation

$$\phi^{ikl} = \left(id_{f_{kl}} * \phi^{ijk} \right)^{-1} \underline{\alpha} \left(\phi^{jkl} * id_{f_{ij}} \right) \phi^{ijl} \quad .$$

Here $(-)^{-1}$ is the inverse 2-morphism map, and $\underline{\alpha}$ is the natural associativity isomorphism. Observe that the valence of the 2-morphisms on the right hand side is strictly less than the valence of ϕ^{ikl} . Hence all 2-morphisms are determined by those of valence 1. \square

Let us prove the theorem:

Proof: It is enough to show that for all n the space of n -simplices, $(\Delta 2\mathcal{C})_n$ is a LEC space.

2Mor is LEC by assumption **G1**. Observe that both Obj and 1Mor are retracts of 2Mor , where the inclusions are provided by identity maps and the retractions by source maps. Hence, Obj and 1Mor are also LEC spaces.

Now proceed by induction on the simplicial degree n . $(\Delta 2\mathcal{C})_0 = \text{Obj}$ and $(\Delta 2\mathcal{C})_1 = 1\text{Mor}$ are LEC by the observation. Assume that $(\Delta 2\mathcal{C})_{n-1}$ is LEC. We can construct $(\Delta 2\mathcal{C})_n$ from the $(n-1)$ -simplices by iterated pullback, i.e. we have a tower

$$(\Delta 2\mathcal{C})_n = P_n \twoheadrightarrow \cdots \twoheadrightarrow P_1 \twoheadrightarrow P_0 = (\Delta 2\mathcal{C})_{n-1}$$

The steps of the tower corresponds to the entries in the last column of the flag describing n -simplices. P_1 is the pullback of

$$P_0 \xrightarrow{x_{n-1}} \text{Obj} \xleftarrow{s_1} 1\text{Mor} \quad .$$

Here we adjoin the 1-morphism $f_{n-1,n}$ along x_{n-1} . The map s_1 is a Hurewicz fibration by assumption **G3**. Hence P_1 is LEC by theorem 6.3. The space P_k , $k > 1$, is the pullback of

$$P_{k-1} \xrightarrow{f_{n-k+1,n} * f_{n-k,n-1+1}} 1\text{Mor} \xleftarrow{t_2} 2\text{Mor} \quad .$$

Here we adjoin the 2-morphisms $\phi^{n-k,n-k+1,n}$ along $f_{n-k+1,n} * f_{n-k,n-1+1}$. By assumption **G4**, t_2 is a Hurewicz fibration. So P_k is LEC by Heath's theorem. \square

Recall from [Dus02] that a 2-groupoid is a 2-category whose 2-morphisms are isomorphisms, and whose 1-morphisms $x_0 \xrightarrow{x_{01}} x_1$ induce equivalences of categories

$$\begin{aligned} \mathbb{B}(x_{01}, x_2)_* &: \mathbb{B}(x_1, x_2) \rightarrow \mathbb{B}(x_0, x_2) \\ \mathbb{B}(x_0, x_{12})_* &: \mathbb{B}(x_0, x_1) \rightarrow \mathbb{B}(x_0, x_2). \end{aligned}$$

We need to make the equivalence of categories continuous. We first remark that an equivalence of categories is the same as a pair of adjoint equivalences. For this and for detail about adjoint functors, see [ML98, chapter IV]. Our approach is to assume that the four structure maps making up this adjoint equivalence are continuous.

We arrive at the following reasonable set of fibrancy conditions:

- SF1.** $2\mathcal{C}$ is a topological 2-groupoid.
- SF2.** The source and target maps for the 1-morphisms are Serre fibrations.
- SF3.** The maps from 2-cells to 1-cells given by the source and by the target of 2-morphisms are both Serre fibrations.
- SF4.** The function sending a 2-morphism to its inverse is continuous.
- SF5.** The functors $\mathbb{B}(x_{01}, x_2)_*$ and $\mathbb{B}(x_0, x_{12})_*$ depend in a continuous way on their arguments.
- SF6.** Moreover, these functors have adjoints, which also depend continuously on their respective arguments.

$$\begin{aligned}\mathbb{B}(x_{01}, x_2)^* &: \mathbb{B}(x_0, x_2) \rightarrow \mathbb{B}(x_1, x_2) \\ \mathbb{B}(x_0, x_{12})^* &: \mathbb{B}(x_0, x_2) \rightarrow \mathbb{B}(x_0, x_1).\end{aligned}$$

- SF7.** Finally, as unit and co-unit of the adjoint equivalences we can choose natural transformations depending continuously on their arguments:

$$\begin{aligned}\epsilon(x_{01}, x_2) &: \mathbb{B}(x_{01}, x_2)^* \circ \mathbb{B}(x_{01}, x_2)_* \rightarrow \text{Id} \\ \epsilon(x_0, x_{12}) &: \mathbb{B}(x_0, x_{12})^* \circ \mathbb{B}(x_0, x_{12})_* \rightarrow \text{Id} \\ \eta(x_{01}, x_2) &: \text{Id} \rightarrow \mathbb{B}(x_{01}, x_2)_* \circ \mathbb{B}(x_{01}, x_2)^* \\ \eta(x_0, x_{12}) &: \text{Id} \rightarrow \mathbb{B}(x_0, x_{12})_* \circ \mathbb{B}(x_0, x_{12})^*.\end{aligned}$$

Lemma 6.6 *Let $\text{source}(f) = \text{source}(h)$, $\text{target}(f) = \text{source}(g)$ and $\text{target}(g) = \text{target}(h) = z$. The adjunctions defines a homeomorphism*

$$X = \{f, g, h, \phi \mid \phi: h \Rightarrow B(f, z)_*(g)\} \rightarrow \{f, g, h, \psi \mid \psi: B(f, z)^*h \Rightarrow g\} = Y$$

Proof: The topology of the spaces are given as subsets of $(1\text{-cells})^3 \times (2\text{-cells})$. The adjunctions define inverse bijections, so we only have to check that they are continuous. But this follows from the continuity of the unit and co-unit. For instance, we see that ψ depends continuously on f, g, h, ϕ since

$$\psi: B(f, z)^*h \xrightarrow{B(f, z)^*(\phi)} B(f, z)^*B(f, z)_*(g) \xrightarrow{\epsilon(f, z)} g.$$

□

We make the definition above because of the following theorem.

Theorem 6.7 *Suppose that $2\mathcal{C}$ satisfies the conditions **SF1**, **SF2**, **SF3**, **SF4**, **SF5**, **SF6**, and **SF7**. Then $2\mathcal{C}$ is sufficiently fibrant, i.e. $\Delta 2\mathcal{C}$ satisfies the topological Kan condition.*

Proof: We have to show that all fill-horn-maps

$$c_k : (\Delta 2\mathcal{C})_n \rightarrow \Lambda_n^k(\Delta 2\mathcal{C})$$

are surjective Serre fibrations. By [Dus02, theorem 8.6] all c_k 's are surjective. It remains to show that they are Serre fibrations. Since $\Delta 2\mathcal{C}$ is 3-coskeletal, the c_k 's are homeomorphisms in dimensions $n > 3$. For $n = 1$ the fill-horn-maps c_0 and c_1 are the source and target maps $1 \text{ Mor} \rightarrow \text{Obj}$, which are Serre fibrations by condition **SF2**. For $n = 3$ the fill-horn-maps ask us to solve the coherence equation

$$(\phi^{123} * f_{01})\phi^{013} = \underline{\alpha}(f_{23} * \phi^{012})\phi^{023}$$

with respect to any one of the ϕ 's given the three others. This can always be done using the inverse of 2-morphisms and the adjoint equivalences of condition **SF7**. The rest of the proof concerns the case $n = 2$:

Using condition **SF4** it follows that the source and target maps of 2-morphisms are surjective Serre fibrations, since they are pull-backs of a surjective Serre fibration.

To finish the proof of the theorem we have to prove that the three maps

$$r_i : 2 \text{Funct}_{str}(\mathcal{D}, 2\mathcal{C}) \rightarrow 2 \text{Funct}_{str}(\mathcal{H}_i, 2\mathcal{C}) \quad \text{for } i = 0, 1, 2,$$

are surjective Serre fibrations.

Case r_1 : Let $(f, g, h, \phi) \in 2 \text{Funct}_{str}(\mathcal{D}, 2\mathcal{C})$ denote the diagram (with obvious compatibilities)

$$\begin{array}{ccc} & \bullet & \\ f \nearrow & & \searrow g \\ \bullet & \xrightarrow{h} & \bullet \\ & \phi \Uparrow & \end{array}$$

We define $F_1(f, g, \phi) = \phi$. Similarly, if f, g are two 1-cells with $\text{source}(g) = \text{target}(f)$, so that they define $(f, g) \in 2 \text{Funct}_{str}(\mathcal{H}_1, 2\mathcal{C})$ we define $G_1(f, g) = B(f, \text{target}(g)) * g$.

There is a pull back diagram

$$\begin{array}{ccc} 2 \text{Funct}_{str}(\mathcal{D}, 2\mathcal{C}) & \xrightarrow{F_1} & 2\text{-cells} \\ \downarrow r_1 & & \downarrow \text{source} \\ 2 \text{Funct}_{str}(\mathcal{H}_1, 2\mathcal{C}) & \xrightarrow{G_1} & 1\text{-cells.} \end{array}$$

Since r_1 is the pull back of a surjective Serre fibration, it is itself a surjective Serre fibration.

Case $r = 0$.

We claim that we can construct a pull back diagram of spaces:

$$\begin{array}{ccc}
 2 \mathit{Funct}_{str}(\mathcal{D}, 2\mathcal{C}) & \xrightarrow{F_0} & 2\text{-cells} \\
 \downarrow r_0 & & \downarrow \text{target} \\
 2 \mathit{Funct}_{str}((\mathcal{H})_1, 2\mathcal{C}) & \xrightarrow{G_0} & 1\text{-cells}.
 \end{array} \tag{1}$$

This suffices to prove the statement of the lemma, since the right hand vertical map is the pull back of a surjective Serre fibration, so it is itself a surjective Serre fibration.

The space $2 \mathit{Funct}_{str}(\mathcal{D}, 2\mathcal{C})$ can be described as triples of 1-morphism (f, g, h) together with a 2-cell ϕ satisfying the compatibility conditions of the space X in lemma 6.6. By this lemma, this space is homeomorphic to the space Y . So we have to construct a pull back diagram of spaces

$$\begin{array}{ccc}
 \{f, g, h, \psi \mid \psi: B(f, \text{target}(h))^*h \Rightarrow g\} & \xrightarrow{F'_0} & 2\text{-cells} \\
 \downarrow (f, g, h, \psi) \mapsto (f, h) & & \downarrow \text{target} \\
 2 \mathit{Funct}_{str}((\mathcal{H})_1, 2\mathcal{C}) & \xrightarrow{G_0} & 1\text{-cells}.
 \end{array} \tag{2}$$

But this is not so hard. We define

$$\begin{aligned}
 F'_0(f, g, h, \psi) &= \psi \\
 G_0(f, h) &= B(f, \text{target}(h))^*(h).
 \end{aligned}$$

Case $r = 0$. This is similar to the case $r = 0$, using the pair of adjunctions $B(\text{source}(h), g)_*$, $B(\text{source}(h), g)^*$. \square

7 Concordance theory

Fix a simplicial space Z_\bullet . In this section we will define the notion of a Z_\bullet -bundle over a topological space X , and study its homotopy properties up to concordance. In the case $Z_\bullet = \Delta 2\mathcal{C}$ this recovers the notion of charted $2\mathcal{C}$ -bundles.

Definition 7.1 *A Z_\bullet -bundle \mathcal{E} over X consists of an ordered open cover \mathcal{U} together with a simplicial map $\phi: U_\bullet \rightarrow Z_\bullet$. Here U_\bullet is the ordered Čech complex of \mathcal{U} . We say that \mathcal{E}_0 and \mathcal{E}_1 over X are concordant if there exists*

a Z_\bullet -bundle \mathcal{E} over $X \times I$ such that the restrictions to $X \times \{0\}$ and $X \times \{1\}$ give \mathcal{E}_0 and \mathcal{E}_1 respectively. Concordance is an equivalence relation. Let $\text{Con}(X, Z_\bullet)$ be the set of all concordance classes of Z_\bullet -bundles over X . The class in $\text{Con}(X, Z_\bullet)$ corresponding to \mathcal{E} will be written as $[\mathcal{E}]$.

Let A be a subspace of X and fix some Z_\bullet -bundle \mathcal{E}_A over A . We say that \mathcal{E} over X is *relative to A* if the restriction $\mathcal{E}|_A$ equals \mathcal{E}_A . Now let Y be the quotient of $X \times I$ where we collapse fibers over points in A . Now, if \mathcal{E}_0 and \mathcal{E}_1 over A are relative to A , then they are *concordant relative to A* if there exists an $\mathcal{E}' = (\mathcal{U}', \phi')$ over Y relative to A restricting to \mathcal{E}_0 and \mathcal{E}_1 on $X \times \{0\}$ and $X \times \{1\}$. We denote the set of such relative concordance classes by $\text{Con}(X, Z_\bullet)_{\text{rel } A}$.

Lemma 7.2 *Assume that A is a subspace of X . Fix \mathcal{E}_A over A . Then the inclusion $i_0 : X \times \{0\} \rightarrow Y = X \times I / \sim$ induces a bijection*

$$\text{Con}(Y, Z_\bullet)_{\text{rel } A} \xrightarrow{\cong} \text{Con}(X, Z_\bullet)_{\text{rel } A} .$$

Proof: By the projection $Y \rightarrow X$ we have that $\text{Con}(X, Z_\bullet)_{\text{rel } A}$ is a retract of $\text{Con}(Y, Z_\bullet)_{\text{rel } A}$. Hence i_0 is surjective. Now assume that the Z_\bullet -bundles \mathcal{E}_0 and \mathcal{E}_1 over Y restrict to same the element in $\text{Con}(X, Z_\bullet)_{\text{rel } A}$. Gluing together \mathcal{E}_0 , \mathcal{E}_1 and the concordance over X relative to A , we get a Z_\bullet -bundle over

$$(X \times I \times \{0\} \cup X \times I \times \{1\} \cup X \times \{0\} \times I) / \sim ,$$

where \sim means that we identify points over A . Pulling back using a retraction $(X \times I \times I) / \sim \rightarrow (X \times I \times \{0\} \cup X \times I \times \{1\} \cup X \times \{0\} \times I) / \sim$, we get a concordance in $\text{Con}(Y, Z_\bullet)_{\text{rel } A}$ between \mathcal{E}_0 and \mathcal{E}_1 . \square

Corollary 7.3 *$\text{Con}(X, Z_\bullet)_{\text{rel } A}$ is a homotopy functor of X relative to A .*

Lemma 7.4 *Let $A \xrightarrow{i_A} B \xrightarrow{i_B} X$ be closed cofibrations. Fix \mathcal{E}_B over B , and let $\mathcal{E}_A = i_A^* \mathcal{E}_B$. If \mathcal{E} over X is relative to A and has restriction $i_B^* \mathcal{E}$ concordant to \mathcal{E}_B relative to A , then there exists a Z_\bullet -bundle \mathcal{E}' concordant to \mathcal{E} relative to A such that $i_B^* \mathcal{E}' = \mathcal{E}_B$.*

Proof: Choose a concordance $\mathcal{E}_{B \times I / \sim}$ relative to A between $i_B^* \mathcal{E}$ and \mathcal{E}_B . Gluing \mathcal{E} and $\mathcal{E}_{B \times I / \sim}$ we get a Z_\bullet -bundle $\bar{\mathcal{E}}$ over $X \cup_B B \times I / \sim$. By the homotopy extension lifting property (HELP), there exists a map $j : X \rightarrow X \cup_B B \times I / \sim$ such that in the following diagram

$$\begin{array}{ccc} B & \longrightarrow & X \cup_B \{0\} \times B \times I / \sim \\ \downarrow & \nearrow j & \downarrow \text{retraction of } B \times I / \sim \\ X & \xlongequal{\quad} & X \end{array}$$

the upper triangle commutes and the lower diagram commutes up to homotopy relative to B . Let $\mathcal{E}' = j^*\bar{\mathcal{E}}$. Clearly $i_B^*\mathcal{E}' = \mathcal{E}_B$. Moreover, j is homotopic to the inclusion $i_X : X \rightarrow X \cup_B B \times I / \sim$ relative to A . Hence,

$$[\mathcal{E}'] = [j^*\bar{\mathcal{E}}] = [i_X^*\bar{\mathcal{E}}] = [\mathcal{E}]$$

in $\text{Con}(X, Z_\bullet)_{\text{rel } A}$. □

Corollary 7.5 *Con($-$, Z_\bullet) is half exact in the following sense: Let $A \xrightarrow{i_A} B \hookrightarrow X$ be closed cofibrations, and fix some Z_\bullet -bundle \mathcal{E}_B over B . Let $\mathcal{E}_A = i_A^*\mathcal{E}_B$. Then the following is an exact sequence of sets*

$$\text{Con}(X, Z_\bullet)_{\text{rel } B} \rightarrow \text{Con}(X, Z_\bullet)_{\text{rel } A} \rightarrow \text{Con}(B, Z_\bullet)_{\text{rel } A} \quad .$$

Proposition 7.6 *If (X, A) is a relative CW-pair, then*

$$\text{Con}(X, Z_\bullet)_{\text{rel } A} \xrightarrow{\cong} \lim_k \text{Con}(X^k, Z_\bullet)_{\text{rel } A}$$

is a bijection. Here X^k is the k -skeleton of X .

Proof: It is enough to prove surjectivity. Fix some \mathcal{E}_A over A , and consider

$$\text{Con}(X, Z_\bullet)_{\text{rel } A} \rightarrow \lim_k \text{Con}(X^k, Z_\bullet)_{\text{rel } A} \quad .$$

An element of the limit is represented by a sequence of $[\mathcal{E}_k] \in \text{Con}(X^k, Z_\bullet)_{\text{rel } A}$ such that the restriction of $[\mathcal{E}_k]$ to X^{k-1} equals the class $[\mathcal{E}_{k-1}]$. We will now construct a sequence \mathcal{E}'_k over X^k such that the restriction of \mathcal{E}'_k to X^{k-1} is exactly equal to \mathcal{E}'_{k-1} , and $[\mathcal{E}'_k] = [\mathcal{E}_k]$. This is done by induction, we start by taking $\mathcal{E}'_0 = \mathcal{E}_0$, and lemma 7.4 provides the induction step. To conclude we see that $\text{colim}_k \mathcal{E}'_k$ is a Z_\bullet -bundle over X mapping to the original sequence of $[\mathcal{E}_k]$'s. □

Proposition 7.7 (Gluing) *Given $B \xleftarrow{f} A \xrightarrow{i} X$, where i is a closed cofibration. Define Y to be the pushout. Fix a Z_\bullet -bundle \mathcal{E}_B over B , and let \mathcal{E}_A be $f^*\mathcal{E}_B$. Then the natural map induces a bijection*

$$\text{Con}(Y, Z_\bullet)_{\text{rel } B} \xrightarrow{\cong} \text{Con}(X, Z_\bullet)_{\text{rel } A} \quad .$$

Proof: If we assume that also f is a closed cofibration the result is obvious – one can then glue and restrict the Z_\bullet -bundles. For arbitrary f , let M be the mapping cylinder of f and let H be the pushout of $M \leftarrow A \rightarrow X$. Observe that the pair (H, M) is homotopy equivalent to (Y, B) . By homotopy invariance of $\text{Con}(-, Z_\bullet)$ the general case follows. □

To handle technicalities later, we need the following:

Definition 7.8 Let A be a subspace of X . Let \mathcal{E} be a Z_\bullet -bundle consisting of the ordered open cover \mathcal{U} and the simplicial map $\phi_\bullet : U_\bullet \rightarrow Z_\bullet$. Let \mathcal{J} be the indices α such that U_α intersects A . We say that \mathcal{E} is collared if there exists an open neighborhood V of A such that $\alpha \in \mathcal{J}$ whenever U_α intersects the closure of V , and there also exists a retraction $r : \bigcup_{\alpha \in \mathcal{J}} U_\alpha \rightarrow A$ such that $r(U_\alpha) = U_\alpha \cap A$ and for every chain $\alpha_0 < \alpha_1 < \dots < \alpha_k$ the following diagram commutes:

$$\begin{array}{ccc} U_{\alpha_0 \alpha_1 \dots \alpha_k} & \xrightarrow{r} & U_{\alpha_0 \alpha_1 \dots \alpha_k} \cap A \\ & \searrow \phi_k & \swarrow \phi_k \\ & & Z_k \end{array} \quad .$$

Lemma 7.9 If $A \hookrightarrow X$ is a closed cofibration, then every Z_\bullet -bundle is concordant relative to A to a collared Z_\bullet -bundle.

Proof: We are given some \mathcal{E} over X subordinate to the ordered open cover \mathcal{U} . Let \mathcal{E}_A be the restriction to A . Using cofibrancy we can take a map $f : X \rightarrow X \cup_{A \times \{0\}} A \times I$ homotopic to the inclusion of X such that $f(a) = (a, 1)$ for all $a \in A$. Now define a new open cover \mathcal{U}' over $X \cup A \times I$ having the following open sets:

- for each $U_\alpha \in \mathcal{U}$ the set $U_\alpha \cup (U_\alpha \cap A \times [0, \frac{2}{3}))$ lies in \mathcal{U}' , and
- for each $U_\alpha \in \mathcal{U}$ such that $U_\alpha \cap A$ is non-empty the set $U_\alpha \cap A \times (\frac{1}{3}, 1]$ lies in \mathcal{U}' .

This open cover can be ordered such that U'_\bullet restricted to $A \times I$ has the homotopy type of simplicial cross product of $U_\bullet \cap A$ and the simplicial 1-simplex Δ^1_\bullet . Now define a Z_\bullet -bundle \mathcal{E}' over $X \cup A \times I$ subordinate to the cover \mathcal{U}' by gluing \mathcal{E} and $\mathcal{E}_A \times \Delta^1_\bullet$. The pullback $f^* \mathcal{E}'$ is concordant relative to A to \mathcal{E} , and furthermore $f^* \mathcal{E}$ is collared. The retraction r comes from the projection $A \times (\frac{1}{3}, 1] \rightarrow A$, and V can be defined as $f^{-1}(A \times (\frac{3}{4}, 1])$. \square

Example 7.10 Wirth and Stasheff, [WS06], describe locally homotopy trivial fibrations with fiber F . It seems plausible that we can define a simplicial space with k -simplexes, Z_k , the space of all homotopy coherent functors from $[k]$ to the topological monoid of homotopy equivalences of F , $H(F)$. In that case a Z_\bullet -bundle should be the same as a homotopy transition cocycle, as given in [WS06, definition 2.5]. Moreover, if Z_\bullet is good, then our theorem 8.1 shows that these fibrations are classified by $|Z_\bullet|$. Furthermore, $|Z_\bullet|$ should be weakly equivalent to $BH(F)$ for reasonable F .

8 Proof of theorem 3.4

In this section let Z_\bullet be any simplicial space. Through a series of lemmas we will establish the following theorem:

Theorem 8.1 *Assume that X has the homotopy type of a CW-complex, and that Z_\bullet is good. Then geometric realization induces a bijection $\text{Con}(X, Z_\bullet) \xrightarrow{\cong} [X, |Z_\bullet|]$.*

Being careful with the definition of Z_\bullet -bundle, we get a sheaf of sets, \mathcal{F} . Thus it makes sense to compare the theorem above to [MW04, proposition A.1.1]. We make stronger assumptions, but our classifying space $|Z_\bullet|$ is more accessible than their $|\mathcal{F}|$.

Of course we have the case $Z_\bullet = \Delta 2\mathcal{C}$ in mind, and when $2\mathcal{C}$ is a good topological 2-category, this condition is satisfied. Moreover, theorem 6.4 provide reasonable criteria for this. Thus we get a proof of the theorem 3.4.

In this section we shall prove the theorem above under the extra condition that Z_\bullet satisfies the topological Kan condition. However, in the next section we will use Quillen's small object argument to generalize the theorem to its stated form, see 9.10.

We are assuming that our base space X has the homotopy type of a CW-complex. In the relative case we assume that $A \hookrightarrow X$ is a closed cofibration, and that both A and X are homotopic to CW-complexes. As explained above, results of [DI04] give a natural weak equivalence $X \xleftarrow{\cong} |U_\bullet|$ for any ordered open cover \mathcal{U} . Therefore, geometric realization gives a map

$$\text{Con}(X, Z_\bullet)_{\text{rel } A} \rightarrow [X, |Z_\bullet|]_{\text{rel } A} \quad .$$

We want to prove that this map is a bijection. The first step of the proof is to reduce to the case where the base space X is a disk and A its boundary.

Lemma 8.2 *If each $\text{Con}(D^n, Z_\bullet)_{\text{rel } S^{n-1}} \rightarrow [D^n, |Z_\bullet|]_{\text{rel } S^{n-1}}$ is a surjection, $n \geq 0$, then $\text{Con}(X, Z_\bullet) \rightarrow [X, |Z_\bullet|]$ is a bijection.*

Proof: Surjectivity for $X \times I$ relative to $X \times \{0, 1\}$ yields injectivity in the absolute case. Hence it is enough to show that $\text{Con}(X, Z_\bullet)_{\text{rel } A} \rightarrow [X, |Z_\bullet|]_{\text{rel } A}$ is surjective.

Since $\text{Con}(-, Z_\bullet)$ and $[-, |Z_\bullet|]$ both are homotopy functors, we can assume that (X, A) is a relative CW-pair. By lemma 7.6 and the corresponding result for $[-, |Z_\bullet|]$, it is enough to show surjectivity for each skeleton X^k of X .

Now consider the diagram

$$\begin{array}{ccc}
\mathrm{Con}(X^k, Z_\bullet)_{\mathrm{rel} X^{k-1}} & \xrightarrow{\textcircled{3}} & [X^k, |Z_\bullet|]_{\mathrm{rel} X^{k-1}} \\
\downarrow & & \downarrow \\
\mathrm{Con}(X^k, Z_\bullet)_{\mathrm{rel} A} & \xrightarrow{\textcircled{2}} & [X^k, |Z_\bullet|]_{\mathrm{rel} A} \\
\downarrow & & \downarrow \\
\mathrm{Con}(X^{k-1}, Z_\bullet)_{\mathrm{rel} A} & \xrightarrow{\textcircled{1}} & [X^{k-1}, |Z_\bullet|]_{\mathrm{rel} A}
\end{array} .$$

The columns are exact by corollary 7.5, and the corresponding result for homotopy classes of maps. $\textcircled{1}$ is surjective by induction hypothesis. Hence, $\textcircled{2}$ is surjective, provided that we can show that $\textcircled{3}$ also is. By proposition 7.7, this follows when $\mathrm{Con}(D^k, Z_\bullet)_{\mathrm{rel} S^{k-1}} \rightarrow [D^k, |Z_\bullet|]_{\mathrm{rel} S^{k-1}}$ is surjective. \square

Let us introduce some notation in order to handle the technical difficulties: Let D be the functor that assigns to the simplicial space Z_\bullet the simplicial set with k -simplices the set of all maps $\sigma : \Delta^k \rightarrow Z_k$. Thus $D(Z_\bullet) = \delta \mathrm{Sing}_\bullet Z_\bullet$. We also have a functor T which sends Z_\bullet to the simplicial space whose space of k -simplices is $\mathrm{Top}(\Delta^k, Z_k)$. Notice that there are natural simplicial maps

$$Z_\bullet \rightarrow T(Z_\bullet) \leftarrow D(Z_\bullet) .$$

In order to complete the proof of theorem 8.1, we look at the following diagram:

$$\begin{array}{ccc}
\mathrm{Con}(D^n, T(Z_\bullet))_{\mathrm{rel} S^{n-1}} & \xrightarrow{\textcircled{3}} & [D^n, |T(Z_\bullet)|]_{\mathrm{rel} S^{n-1}} \\
\textcircled{4} \uparrow & & \uparrow \textcircled{2} \\
\mathrm{Con}(D^n, Z_\bullet)_{\mathrm{rel} S^{n-1}} & \xrightarrow{\textcircled{1}} & [D^n, |Z_\bullet|]_{\mathrm{rel} S^{n-1}}
\end{array} .$$

Our aim is to show that $\textcircled{1}$ is a surjection. Assume that Z_\bullet is good and satisfies the topological Kan condition. Then lemma 8.3 below implies that $\textcircled{2}$ is a bijection. Lemma 8.7 below says that $\textcircled{3}$ is surjective. At last theorem 9.9, from the next section, says that $\textcircled{4}$ also is surjective. Modulo showing these lemmas, we have completed the proof.

Lemma 8.3 *After taking geometric realization, we get that $|Z_\bullet|$ is a strong deformation retract of $|T(Z_\bullet)|$.*

Proof: Given a point $\mathbf{t} = (t_0, \dots, t_n)$ in the topological n -simplex Δ^n , define the contraction at \mathbf{t} to be the map $c_{\mathbf{t}} : \Delta^n \times I \rightarrow \Delta^n$ given by

$$c_{\mathbf{t}}(u_0, \dots, u_n; s) = ((1-s)u_0 + st_0, (1-s)u_1 + st_1, \dots, (1-s)u_n + st_n) .$$

Write $c_{\mathbf{t},s} : \Delta^n \rightarrow \Delta^n$ when fixing some $s \in I$. Suppose that $\psi : [m] \rightarrow [n]$ is an order preserving map. Let $\psi_* : \Delta^m \rightarrow \Delta^n$ be the induced map on topological simplexes. Then the following diagram commutes:

$$\begin{array}{ccc} \Delta^m \times I & \xrightarrow{\psi_* \times id} & \Delta^n \times I \\ c_{\mathbf{t}} \downarrow & & \downarrow c_{\psi_*(\mathbf{t})} \\ \Delta^m & \xrightarrow{\psi_*} & \Delta^n \end{array}$$

Let K be a topological space and define $S(K)_\bullet$ to be the simplicial space having n -simplices $\mathcal{Top}(\Delta^n, K)$. Observe that the subspace of $|S(K)_\bullet|$ spanned by the constant maps $\sigma : \Delta^n \rightarrow K$ is canonically identified with K . We now claim that K is a strong deformation retraction of $|S(K)_\bullet|$. To see this define a map

$$H : |S(K)_\bullet| \times I \rightarrow |S(K)_\bullet|$$

as follows:

A point in $|S(K)_\bullet|$ is determined by a pair (σ, \mathbf{t}) where $\sigma : \Delta^n \rightarrow K$ and $\mathbf{t} \in \Delta^n$. Let

$$H(\sigma, \mathbf{t}; s) = (\sigma c_{\mathbf{t},s}, \mathbf{t}) \quad .$$

Furthermore, if $\psi : [m] \rightarrow [n]$ is order preserving, $\sigma : \Delta^n \rightarrow K$, and $\mathbf{t} \in \Delta^m$, then $(\sigma \psi_*, \mathbf{t})$ and $(\sigma, \psi_* \mathbf{t})$ are identified in the geometric realization $|S(K)_\bullet|$. Computing in the geometric realization, we have

$$H(\sigma \psi_*, \mathbf{t}; s) = (\sigma \psi_* c_{\mathbf{t},s}, \mathbf{t}) = (\sigma c_{\psi_* \mathbf{t},s} \psi_*, \mathbf{t}) = (\sigma c_{\psi_* \mathbf{t},s}, \psi_* \mathbf{t}) = H(\sigma, \psi_* \mathbf{t}; s) \quad .$$

Hence, H is well-defined.

Now let K run through the spaces of k -simplices of Z_\bullet . By naturality of H we get that $|Z_\bullet|$ is a strong deformation retract of $||S(Z_\bullet)_\bullet||$. On the other side, the diagonal of the bisimplicial space $S(Z_\bullet)_\bullet$ is $T(Z_\bullet)$, and it is well known that $||S(Z_\bullet)_\bullet||$ is homeomorphic to $|T(Z_\bullet)|$. Therefore, we see that $|Z_\bullet|$ is a strong deformation retract of $|T(Z_\bullet)|$. \square

Lemma 8.4 *There is a natural map $|D(Z_\bullet)| \rightarrow |Z_\bullet|$, and this map is a weak equivalence provided that Z_\bullet is good, i.e. all degeneracies are cofibrations. Moreover, the diagram*

$$\begin{array}{ccc} |Z_\bullet| & \longleftarrow & |D(Z_\bullet)| \\ & \searrow & \swarrow \\ & |T(Z_\bullet)| & \end{array}$$

commutes up to homotopy.

Proof: The first part is well known, while the second part follows easily from the proof of the previous lemma. \square

Theorem 8.5 (Whitney) *Every open subset of S^{n-1} is a CW-complex.*

Proof: Every proper subset U of S^{n-1} embeds as an open subset of \mathbb{R}^{n-1} . Consider meshes \mathcal{M}_s in \mathbb{R}^{n-1} of cubes with side length 2^{-s} , such that \mathcal{M}_{s+1} is a subdivision of \mathcal{M}_s . Following [Whi34, section 8], we can write U as a union $\bigcup_s K_s$ such that K_s consists of cubes in \mathcal{M}_s , the interiors of the K_s 's are mutually disjoint, and K_s meets only K_{s-1} and K_{s+1} . This is a CW-structure on U . \square

In the lemma 8.6 below, we start out with a simplicial map $U_\bullet \rightarrow T(Z_\bullet)$ and then attempt to improve it. Hence we need explicit descriptions of such maps: Let U_\bullet be a simplicial space with free degeneracies. Then the subspace N_k of U_k consisting of the non-degenerate simplexes is disjoint from the degenerate simplexes. Moreover, N_\bullet forms a presimplicial space, i.e. having all face maps, but without any degeneracy maps. Furthermore, simplicial maps

$$U_\bullet \rightarrow T(Z_\bullet)$$

are in one-to-one correspondence with presimplicial maps

$$N_\bullet \rightarrow T(Z_\bullet) \quad .$$

Recalling the definition of $T(Z_\bullet)$, we see that maps of the latter type can be described by continuous maps $\phi_k : N_k \times \Delta^k \rightarrow Z_k$ such that the following *presimplicial coherence diagrams* commute for all face maps d_i :

$$\begin{array}{ccc}
 & N_k \times \Delta^k & \xrightarrow{\phi_k} & Z_k & . \\
 & \nearrow^{id \times \delta_i} & & \downarrow d_i & \\
 N_k \times \Delta^{k-1} & & & & \\
 & \searrow_{d_i \times id} & & & \\
 & N_{k-1} \times \Delta^{k-1} & \xrightarrow{\phi_{k-1}} & Z_{k-1} &
 \end{array}$$

Lemma 8.6 *Assume that Z_\bullet satisfies the topological Kan condition and \mathcal{U} is a good ordered open cover of S^{n-1} . Then every simplicial map $U_\bullet \rightarrow T(Z_\bullet)$ is topologically homotopic to a map $U_\bullet \rightarrow D(Z_\bullet)$.*

Proof: Recall that the ordered Čech complex U_\bullet has free degeneracies. The space of non-degenerate k -simplices is

$$N_k = \coprod_{\alpha_0 < \dots < \alpha_k} U_{\alpha_0 \dots \alpha_k} \quad ,$$

and each N_k is a CW-complex by theorem 8.5. Moreover, each connected component of N_k is contractible since \mathcal{U} is a good cover. Hence, there exists local contractions $H^k : N_k \times I \rightarrow N_k$, i.e. homotopies such that for $x \in N_k$ we have $H^k(x, 1) = x$, whereas $H^k(x, 0)$ is locally constant.

Let $\phi_k : N_k \times \Delta^k \rightarrow Z_k$ be a collection of maps that represents a simplicial map $U_\bullet \rightarrow T(Z_\bullet)$. By induction on k we will construct maps $\Phi_k : N_k \times I \times \Delta^k \rightarrow Z_k$ such that

$$\Phi_k(x, 1, \mathbf{t}) = \phi_k(x, \mathbf{t}) \text{ for all } x \in N_k \text{ and } \mathbf{t} \in \Delta^k,$$

for $\mathbf{t} \in \Delta^k$ fixed, the expression $\Phi_k(x, s, \mathbf{t})$ is locally constant as a function $N_k \times \left[0, \frac{1}{k+2}\right] \rightarrow Z_k$, and

all presimplicial coherence diagrams for Φ_\bullet commute.

We begin by defining Φ_0 . For $x \in N_0$, $s \in I$ and $1 \in \Delta^0$ we define

$$\Phi_0(x, s, 1) = \begin{cases} \phi_0(H^0(x, 2s - 1), 1) & \text{if } s \geq \frac{1}{2} \\ \phi_0(H^0(x, 0), 1) & \text{if } s \leq \frac{1}{2}. \end{cases}$$

Next, we construct Φ_k from Φ_i , $i < k$. This is done in four steps:

First step. In this step we will extend $\phi_k : N_k \times \{1\} \times \Delta^k \rightarrow Z_k$ to a map

$$\Phi_k : \left(N_k \times \left[\frac{1}{k+1}, 1\right] \times \partial\Delta^k\right) \cup (N_k \times \{1\} \times \Delta^k) \rightarrow Z_k \quad .$$

We will do this by induction over the dimension of proper faces σ of Δ^k . Given a face σ , let I be the subset of $[k]$ such that $n \in I$ if and only if σ is contained in the n 'th $(k-1)$ -face of Δ^k . We can, by the induction hypothesis, construct a diagram

$$\begin{array}{ccc} N_k \times \left(\left[\frac{1}{k+1}, 1\right] \times \partial\sigma \cup \{1\} \times \sigma\right) & \longrightarrow & Z_k \\ \downarrow & & \downarrow c_k^I \\ N_k \times \left[\frac{1}{k+1}, 1\right] \times \sigma & \longrightarrow & C\Lambda_k^I(Z_\bullet) \end{array} \quad ,$$

where the lower map comes from the Φ_i , $i < k$. The vertical map on the left side is a cellular inclusion between CW-complexes, and a homotopy

equivalence. The topological Kan condition says that c_k^I is a Serre fibration. Hence, a lift exists, and we use this to define Φ_k on $N_k \times \left[\frac{1}{k+1}, 1\right] \times \sigma$.

Second step. Let

$$r : \left[\frac{1}{k+1}, 1\right] \times \Delta^k \rightarrow \left(\left[\frac{1}{k+1}, 1\right] \times \partial\Delta^k\right) \cup (\{1\} \times \Delta^k)$$

be a retraction. For $x \in N_k$, $\frac{1}{k+1} \leq s \leq 1$, $\mathbf{t} \in \Delta^k$, we define

$$\Phi_k(x, s, \mathbf{t}) = \Phi_k(r(x, s, \mathbf{t})) \quad ,$$

where the Φ_k on the right side is defined by the previous step.

Third step. Recall that H^k denoted a local contraction of N_k . Choose a function $f : \left[\frac{1}{k+2}, \frac{1}{k+1}\right] \rightarrow I$ such that $f(\frac{1}{k+2}) = 0$ and $f(\frac{1}{k+1}) = 1$. For $x \in N_k$, $\frac{1}{k+2} \leq s \leq 1$, $\mathbf{t} \in \Delta^k$, we now define

$$\Phi_k(x, s, \mathbf{t}) = \begin{cases} \Phi_k(x, s, \mathbf{t}) & \text{if } \frac{1}{k+1} \leq s, \\ \Phi_k(H^k(x, f(s)), \frac{1}{k+1}, \mathbf{t}) & \text{if } \frac{1}{k+2} \leq s \leq \frac{1}{k+1}. \end{cases}$$

Again, the Φ_k on the right comes from the previous step. At this stage, we should verify that the presimplicial coherence diagrams commute for $s \in \left[\frac{1}{k+2}, \frac{1}{k+1}\right]$, i.e. we need to check the equality

$$d_i \Phi_k(x, s, \delta_i \mathbf{t}) = \Phi_{k-1}(d_i x, s, \mathbf{t})$$

for all face maps d_i , $x \in N_k$, s as above and $t \in \Delta^{k-1}$. This is an easy exercise using that equality holds for $s = \frac{1}{k+1}$, and that when fixing \mathbf{t} the function $\Phi_{k-1}(y, s, \mathbf{t})$ is locally constant on $N_{k-1} \times \left[0, \frac{1}{k+1}\right]$.

Final step. The steps above define Φ_k on $N_k \times \left[\frac{1}{k+2}, 1\right] \times \Delta^k$. Now extend the domain to all of $N_k \times I \times \Delta^k$ by

$$\Phi_k(x, s, \mathbf{t}) = \Phi_k(x, \frac{1}{k+2}, \mathbf{t}) \quad , \text{ for } s \leq \frac{1}{k+2}.$$

This completes the construction of all Φ_k 's.

Since $\Phi_k(x, 0, \mathbf{t})$ is locally constant for fixed $\mathbf{t} \in \Delta^k$, it follows that $\Phi_\bullet(-, 0, -)$ corresponds to a simplicial map $U_\bullet \rightarrow D(Z_\bullet)$. This completes the proof of the lemma. \square

Lemma 8.7 *If Z_\bullet is good and satisfies the topological Kan condition, then the map $\text{Con}(D^n, T(Z_\bullet))_{\text{rel } S^{n-1}} \rightarrow [D^n, |T(Z_\bullet)|]_{\text{rel } S^{n-1}}$ is surjective.*

Proof: Fix a $T(Z_\bullet)$ -bundle \mathcal{E}_∂ over S^{n-1} , and let \mathcal{V} denote the corresponding ordered open cover. Since every point in D^n has an arbitrarily small convex neighborhood, it is possible to find a good ordered open cover \mathcal{U} of D^n such that the restriction of \mathcal{U} to S^{n-1} is a refinement of \mathcal{V} . Let \mathcal{U}_∂ denote the restriction of \mathcal{U} to S^{n-1} . It is possible to construct an ordered open cover \mathcal{V}' over $S^{n-1} \times I$ such that the restriction of \mathcal{V}' to $S^{n-1} \times \{0\}$ equals \mathcal{V} , and the restriction of \mathcal{V}' to $S^{n-1} \times [\frac{1}{2}, 1]$ is the product of $[\frac{1}{2}, 1]$ and \mathcal{U}_∂ . Now extend \mathcal{E}_∂ to a $T(Z_\bullet)$ -bundle $\mathcal{E}_{S^{n-1} \times [0, \frac{1}{2}]}$ over $S^{n-1} \times [0, \frac{1}{2}]$ subordinate to the cover \mathcal{V}' . By lemma 8.6 we can assume that $\mathcal{E}_{S^{n-1} \times I}$ restricted to $S^{n-1} \times \{1\}$ is a $D(Z_\bullet)$ -bundle represented by \mathcal{U}_∂ and a simplicial map $\phi : (U_\partial)_\bullet \rightarrow D(Z_\bullet)$.

Now consider the diagram

$$\begin{array}{ccc} [U_\bullet, D(Z_\bullet)]_{\text{rel } \phi} & \xrightarrow{\textcircled{2}} & [D^n, |D(Z_\bullet)|]_{\text{rel } S^{n-1}} \\ \textcircled{1} \downarrow & & \downarrow \textcircled{3} \\ \text{Con}(D^n, T(Z_\bullet))_{\text{rel } S^{n-1}} & \xrightarrow{\textcircled{4}} & [D^n, |T(Z_\bullet)|]_{\text{rel } S^{n-1}} \end{array} .$$

Here $\textcircled{1}$ is given by gluing together an arbitrary map $U_\bullet \rightarrow Z_\bullet$ with the $T(Z_\bullet)$ -bundle $\mathcal{E}_{S^{n-1} \times I}$ along the boundary. By theorem 5.3 $D(Z_\bullet)$ is a Kan complex. Since U_\bullet is good, it follows that $\textcircled{2}$ is a bijection. By lemma 8.4 it follows that $\textcircled{3}$ also is a bijection. Hence, $\textcircled{4}$ is surjective. \square

9 Fibrant replacement

In this section we will show how to construct a fibrant replacement \tilde{Z}_\bullet for any good simplicial space Z_\bullet . Moreover, the natural map $Z_\bullet \rightarrow \tilde{Z}_\bullet$ induces a natural bijection $\text{Con}(X, Z_\bullet) \rightarrow \text{Con}(X, \tilde{Z}_\bullet)$.

An open cover \mathcal{U} is *finite and totally ordered* if it comes with a finite and totally ordered index set \mathcal{I} . Thus we may define $\text{Con}^{\text{finite}}(X, Z_\bullet)$, and correspondingly in the relative case. Over compact spaces there is no loss of generality when using such covers:

Lemma 9.1 *If X is compact and A a neighborhood deformation retract, then the natural map $\text{Con}^{\text{finite}}(X, Z_\bullet)_{\text{rel } A} \rightarrow \text{Con}(X, Z_\bullet)_{\text{rel } A}$ is a bijection for all Z_\bullet .*

Proof: It is sufficient to consider surjectivity when fixing a Z_\bullet -bundle \mathcal{E}_A having a finite and totally ordered open cover. By compactness, any ordered open cover of X has a finite refinement. Moreover, we may choose a total ordering of the indexing set extending the already existing partial ordering.

□

Corollary 9.2 *Let $Z_\bullet \rightarrow W_\bullet$ be a map of simplicial spaces. If $\text{Con}^{\text{finite}}(D^n, Z_\bullet)_{\text{rel } \partial} \rightarrow \text{Con}^{\text{finite}}(D^n, W_\bullet)_{\text{rel } \partial}$ is surjective for all $n \geq 0$, then $\text{Con}(X, Z_\bullet)_{\text{rel } A} \rightarrow \text{Con}(X, W_\bullet)_{\text{rel } A}$ is a bijection for all pairs (X, A) of the homotopy type of a CW-pair.*

Lemma 9.3 *Let W_\bullet be the pushout of $\Delta_\bullet^n \leftarrow \Lambda_\bullet^{n,k} \rightarrow Z_\bullet$. Then the natural map $\text{Con}(X, Z_\bullet) \rightarrow \text{Con}(X, W_\bullet)$ is a bijection.*

Proof: By the above corollary, it is enough to show surjectivity in the relative case

$$\text{Con}^{\text{finite}}(D^m, Z_\bullet)_{\text{rel } S^{m-1}} \rightarrow \text{Con}^{\text{finite}}(D^m, W_\bullet)_{\text{rel } S^{m-1}} \quad .$$

Hence, we attempt to lift a W_\bullet -bundle \mathcal{E} subordinate to a finite and totally ordered open cover \mathcal{U} . We may assume that \mathcal{E} is collared. Let σ_k denote the $(n-1)$ -face of Δ_\bullet^n which is missing in $\Lambda_\bullet^{n,k}$. σ_k is also an isolated point of W_{n-1} . Let N be the number of subsets $S \subseteq \mathcal{I}$ such that $U_S = \bigcup_{\alpha \in S} U_\alpha$ maps to a degenerate simplex over σ_k . Observe that N is finite, and that \mathcal{E} lifts to a Z_\bullet -bundle if $N = 0$. We will now give a construction of a concordant W_\bullet -bundle \mathcal{E}' such that N decreases.

Let r be the largest integer such that there exists a increasing chain $\alpha_0 < \dots < \alpha_r$ of indices such that $U_{\alpha_0 \dots \alpha_r}$ maps to some degenerate simplex over σ_k . This degenerate simplex is described by a surjective order preserving map $\phi : \{\alpha_0, \dots, \alpha_r\} \rightarrow [n-1]$. By the collaring of \mathcal{E} we may assume that none of the U_{α_i} 's meet the boundary $\partial = S^{m-1}$. Define $U_\gamma = U_{\alpha_0 \dots \alpha_r}$. Let β_0, \dots, β_s be all indices β such that $U_\beta \cap U_\gamma$ is non-empty. Without loss of generality, we may assume that the β 's are ordered increasingly. Moreover, if R is $\{\alpha_0, \dots, \alpha_r, \beta\}$, then U_R maps to a degenerate simplex over the non-degenerate n -simplex $\iota \in \Delta_n^n \rightarrow W_n$. Thus we have a diagram

$$\begin{array}{ccc} \{\alpha_0 < \dots < \alpha_r\} & \longrightarrow & \{\alpha_0 < \dots < \alpha_j < \beta_0 < \dots < \beta_s < \alpha_{j+1} < \dots < \alpha_r\} \\ \phi \downarrow & & \downarrow \hat{\phi} \\ [n-1] & \xrightarrow{\sigma_k} & [n] \end{array} \quad .$$

Here $\hat{\phi}$ must map each β_i to k , otherwise we would violate our assumption on r . Let us now define a new ordered open covering \mathcal{U}' by adding one additional open set, namely $U_\gamma = V$ with $\alpha_j < \gamma < \beta_0$. We can extend the W_\bullet -bundle to \mathcal{U}' by declaring that U_R should map to the simplex of Δ_\bullet^n given by $\theta : R \rightarrow [n]$ where $\theta(\gamma) = k$ and otherwise θ agrees with $\hat{\phi}$.

Now take a partition of unity $\{\psi_\alpha\}$ subordinate to the original cover \mathcal{U} . We proceed to define a refinement \mathcal{U}'' of $\mathcal{U}' = \mathcal{U} \cup \{U_\gamma\}$. For all $\beta \in \mathcal{I} \cup \{\gamma\}$ not of the form α_i we set $U''_\beta = U_\beta$. Let U''_{α_i} be the set

$$\left\{ x \mid \psi_{\alpha_i}(x) > \frac{1}{r} (\psi_{\alpha_0}(x) + \cdots + \psi_{\alpha_{i-1}}(x) + \psi_{\alpha_{i+1}}(x) + \cdots + \psi_{\alpha_r}(x)) \right\}.$$

We check that this is a cover: Take a point x . If all $\psi_{\alpha_i}(x) = 0$, then x lies in some $U''_\alpha = U_\alpha$ for $\alpha \in \mathcal{I} \setminus \{\alpha_0, \dots, \alpha_r\}$. If $\psi_{\alpha_0}(x) = \psi_{\alpha_1}(x) = \cdots = \psi_{\alpha_r}(x) > 0$, then x lies in U_γ . Otherwise, there exists an j such that $\psi_{\alpha_j}(x) \geq \psi_{\alpha_i}(x)$ for all i , and the inequality is strict at least for one i . This implies that $x \in U''_{\alpha_j}$. Moreover, the intersection $\bigcap_i U''_{\alpha_i}$ is empty. This can be seen by taking the sum over all the inequalities. Thus we have a new W_\bullet -bundle with reduced integer N . \square

Lemma 9.4 *Let N_\bullet be the non-degenerate part of the Čech complex corresponding to a finite and totally ordered open covering. The following properties are satisfied:*

- *There exists an integer K so that $N_k = \emptyset$ for $k > K$.*
- *If $d_i(x) = d_j(y) \in N_k$, $i < j$ there is an $z \in N_{k+2}$ such that $y = d_i(z)$ and $x = d_{j+1}(z)$.*
- *The obvious map*

$$\coprod_{0 \leq i \leq k+1} N_{k+1} \xrightarrow{\cup_i d_i} \cup_i d_i(N_{k+1}) \subseteq N_k$$

is a quotient map.

Proof: Chose K to be the number of elements in the cover. The next two statements are general properties of the non-degenerate part of a Čech complex.

Addressing the last statement, we consider a subset V of N_k such that all $d_i^{-1}(V)$ are open in N_{k+1} . We have to show that $V \cap (\cup_i d_i(N_{k+1}^\epsilon))$ is open. We can assume that V is a subset of some $U_{\alpha_0 \dots \alpha_k}$. Notice that all $V \cap U_\beta$ are open for $\beta \neq \alpha_i$. The result follows since $\cup_i d_i(N_{k+1}^\epsilon)$ equals the finite union $\bigcup_{\beta \neq \alpha_i} (V \cap U_\beta)$. \square

The following lemma plays the role of “a homotopy extension property” for the inclusion $\cup_i d_i(N_{k+1}) \hookrightarrow N_k$. In a vague sense this is dual to Segal’s definition of a good simplicial space, [Seg74].

Lemma 9.5 *Let \mathcal{U} be a finite and totally ordered open cover, let N_\bullet be the non-degenerate part of its ordered Čech complex, let Z be any space, and let $G : (\cup_i d_i(N_{k+1})) \times I \rightarrow Z$ be a homotopy which agrees with a map $g : N_k \rightarrow Z$ for $0 \in I$. Then there exists a refinement \mathcal{U}^ϵ of \mathcal{U} and a homotopy $G^\epsilon : N_k^\epsilon \times I \rightarrow Z$ such that G^ϵ agrees with G on $(\cup_i d_i(N_{k+1}^\epsilon)) \times I$ and with g on $N_k^\epsilon \times \{0\}$. Moreover, if $t \mapsto G(x, t)$ is constant for some $x \in \cup_i d_i(N_{k+1}^\epsilon)$, then $t \mapsto G^\epsilon(x, t)$ is also constant.*

Proof: Choose a partition of unity ψ_α subordinate to the finite and totally ordered cover \mathcal{U} . For $\epsilon > 0$ let \mathcal{U}^ϵ be the collection of all open sets $\psi_\alpha^{-1}(\epsilon, 1]$. If ϵ is small, $\epsilon < \frac{1}{|\mathcal{I}|}$, where $|\mathcal{I}|$ is the number of open sets in \mathcal{U} , then \mathcal{U}^ϵ is again a cover.

Now define a map $\psi : N_k \rightarrow I$ as follows: On the intersection $U_{\alpha_0 \dots \alpha_k}$ let ψ be the sum $\sum_{\beta \neq \alpha_i} \psi_\beta$. Notice that the support of ψ is contained in $\cup_i d_i(N_{k+1})$, and that $\psi^{-1}(\epsilon, 1]$ is $\cup_i d_i(N_{k+1}^\epsilon)$.

Define a homotopy $\hat{G}^\epsilon : N_k \times I \rightarrow Z$ by the formula:

$$\hat{G}^\epsilon(x, t) = \begin{cases} g(x) & \text{if } x \notin \cup_i d_i(N_{k+1}), \\ G(x, t) & \text{if } x \in \cup_i d_i(N_{k+1}) \text{ and } \psi(x) \geq \epsilon, \text{ and} \\ G(x, \frac{\psi(x)}{\epsilon} t) & \text{if } x \in \cup_i d_i(N_{k+1}) \text{ and } \psi(x) \leq \epsilon. \end{cases}$$

Let G^ϵ be the restriction of \hat{G}^ϵ to $N_k^\epsilon \times I$. □

Theorem 9.6 *Let $i : Z_\bullet \rightarrow W_\bullet$ be a map of simplicial spaces with the property that each $i_n : Z_n \rightarrow W_n$ is the inclusion of a strong deformation retract. Suppose that U_\bullet is the Čech complex of a finite and totally ordered open cover. Let $h : U_\bullet \rightarrow W_\bullet$ be a simplicial map. There exist a refinement \mathcal{U}^ϵ of \mathcal{U} together with simplicial maps $g : U_\bullet^\epsilon \rightarrow Z_\bullet$ and $F : U_\bullet^\epsilon \times I \rightarrow W_\bullet$ with the properties that $F(x, 1) = i(g(x))$ and $F(x, 0) = h(x)$. Moreover, if h takes $x \in U_n^\epsilon$ to Z_n , then $t \mapsto F_n(x, t)$ is constant.*

Proof: It is enough to consider the non-degenerate part N_\bullet of U_\bullet , i.e. we can reformulate the question in the category of presimplicial spaces.

We construct these by downward induction on the simplicial degree. Assume that we have constructed a refinement \mathcal{U}^{ϵ_n} of \mathcal{U} and a map in the category of presimplicial spaces.

$$F^n : N_\bullet^{\epsilon_n} \times I \rightarrow W_\bullet$$

such that $F^n(x, 0) = h(x)$ and $F^n(N_k^{\epsilon_n} \times \{1\}) \subset Z_k$ for all $k > n$. To complete the inductive step and define F^{n-1} , it suffices to construct a refinement $\mathcal{U}^{\epsilon_{n-1}}$ of \mathcal{U}^{ϵ_n} and a map of presimplicial spaces $G : N_\bullet^{\epsilon_{n-1}} \times I \rightarrow W_\bullet$ with the following properties:

- $G_k(x, 0) = F_k^n(x, 1)$.
- $G_k(x, t) = F_k^n(x, 1)$ if $k > n$.
- $G_n(x, 1) \in Z_n \subset W_n$.

Then we obtain F^{n-1} by composing the homotopy F^n with the homotopy G

For $k > n$, the conditions already define G_k . We define $G_n : N_n^{\epsilon_n} \times I \rightarrow W_n$ as follows: Let $H : W_n \times I \rightarrow W_n$ be the deformation retraction into Z_n , and let $G_n(x, t) = H(F_n^n(x, 1), t)$.

Since $F_{n+1}(N_{n+1}^{\epsilon_n} \times \{1\}) \subset Z_{n+1}$, for each i , $0 \leq i \leq n+1$, we get a commutative diagram

$$\begin{array}{ccc} N_{n+1}^{\epsilon_n} \times I & \xrightarrow{d_i} & N_n^{\epsilon_n} \times I \\ \downarrow G_{n+1} & & \downarrow G_n \\ Z_{n+1} & \xrightarrow{d_i} & Z_n \xrightarrow{i_n} W_n \end{array}$$

Thus G clearly satisfies the conditions, except for that we have not yet defined G_k for $k < n$. The conditions these maps have to satisfy are that the maps G_k , for $0 \leq k$, define a presimplicial map, and the compatibility condition that $G_k(x, 0) = F_k^n(x, 1)$. Everything else is already taken care of. We define G_k by downward induction on k , using lemma 9.5.

To be precise, assume that we have descended to stage $k+1$. We have defined G_{k+1} , and want to proceed to define G_k . At each stage we have to shrink our cover. Thus G_{k+1} is defined on $N_{k+1}^{\epsilon_n, k+1} \times I$. Here $N_{\bullet}^{\epsilon_n, k+1}$ corresponds to a refinement $\mathcal{U}^{\epsilon_n, k+1}$ of $\mathcal{U}^{\epsilon_n, k+2}$. The G_k we seek should be compatible with F_k^n , and make the following diagram commute for each i , $0 \leq i \leq k+1$:

$$\begin{array}{ccc} N_{k+1} \times I & \xrightarrow{d_i} & N_k \times I \\ \downarrow G_{k+1} & & \downarrow G_k \\ W_{k+1} & \xrightarrow{d_i} & W_k \end{array}$$

Because of the second statement of lemma 9.4, there is a well defined set map $\tilde{G}_k : \cup_{0 \leq i \leq k+1} d_i(N_{k+1}^{\epsilon_n, k+1} \times I) \rightarrow W_k$ such that the following diagram commutes:

$$\begin{array}{ccc} N_{k+1}^{\epsilon_n, k+1} \times I & \xrightarrow{d_i} & \cup_{0 \leq i \leq k+1} d_i(N_{k+1}^{\epsilon_n, k+1} \times I) \\ \downarrow G_{k+1} & & \downarrow \tilde{G}_k \\ W_{k+1} & \xrightarrow{d_i} & W_k \end{array}$$

According to the last statement of lemma 9.4, the map \tilde{G}_k is a continuous map. By lemma 9.5, we can find a refinement $\mathcal{U}^{\epsilon_{n,k}}$ of the cover $\mathcal{U}^{\epsilon_{n,k+1}}$, such that an extension in the diagram

$$\begin{array}{ccc}
\cup_i d_i(N_{k+1}^{\epsilon_{n,k}} \times \{0\}) & \longrightarrow & \cup_i d_i(N_{k+1}^{\epsilon_{n,k}} \times I) \\
\downarrow & & \downarrow \\
N_k^{\epsilon_{n,k}} \times \{0\} & \longrightarrow & N_k^{\epsilon_{n,k}} \times I \\
& \searrow^{(x,0) \mapsto F_k^n(x,1)} & \searrow^{G_k} \\
& & W_k
\end{array}$$

exists. This completes the inductive step, and the proof of everything except the last part. However, tracing the proof it is easily seen that all homotopies fix points $x \in U_n^\epsilon$ having $h_n(x) \in Z_n$. \square

Corollary 9.7 *Assume that $i : Z_\bullet \rightarrow W_\bullet$ is as above. Let $A \subseteq X$ be a closed cofibration, and assume that (X, A) has the homotopy type of a CW-complex. Then*

$$\text{Con}(X, Z_\bullet)_{\text{rel } A} \rightarrow \text{Con}(X, W_\bullet)_{\text{rel } A}$$

is a bijection.

Proof: By corollary 9.2 it is sufficient to consider surjectivity in the relative case

$$\text{Con}^{\text{finite}}(D^m, Z_\bullet)_{\text{rel } S^{m-1}} \rightarrow \text{Con}^{\text{finite}}(D^m, W_\bullet)_{\text{rel } S^{m-1}} \quad .$$

Here we have fixed a Z_\bullet -bundle \mathcal{E}_∂ over S^{m-1} . Let \mathcal{U}_∂ be the cover of S^{m-1} corresponding to \mathcal{E}_∂ . Without loss of generality it is enough to lift a W_\bullet -bundle \mathcal{E}^W collared with respect to $i_*\mathcal{E}_\partial$. The theorem above constructs a lifting \mathcal{E}^Z . However, \mathcal{E}^Z agrees with \mathcal{E}_∂ only on a refinement \mathcal{U}_∂^ξ of \mathcal{U}_∂ . This is not a problem, since we can extend \mathcal{E}^Z by gluing onto it the collar of \mathcal{E}_∂ . \square

Corollary 9.8 *Let $A \subseteq X$ be a closed cofibration, and assume that (X, A) has the homotopy type of a CW-complex. Then the functor $\text{Con}(X, Z_\bullet)_{\text{rel } A}$ is invariant under topological homotopy of Z_\bullet .*

Proof: The map $Z_\bullet \xrightarrow{i_0} Z_\bullet \times I$ is in each degree a strong deformation retraction. \square

Theorem 9.9 *Let $A \subseteq X$ be a closed cofibration, and assume that (X, A) has the homotopy type of a CW-complex. Consider $f : Z_\bullet \rightarrow W_\bullet$, a map of simplicial spaces such that each f_n is a homotopy equivalence. Then f induces a bijection*

$$f_* : \text{Con}(X, Z_\bullet)_{\text{rel } A} \rightarrow \text{Con}(X, W_\bullet)_{\text{rel } A}$$

Proof: Let M_\bullet be the degreewise mapping cylinder of f . There are inclusions of degreewise strong deformation retracts $i^Z : Z_\bullet \subset M_\bullet$ and $i^W : W_\bullet \subset M_\bullet$. Moreover, the map $i^W \circ f$ is homotopic to i^Z . The theorem holds for i^W and i^Z by corollary 9.7. By corollary 9.8 i^Z and $i^W \circ f$ induce the same map of concordance classes. Hence, we obtain a commutative diagram, which proves the claim for f_* .

$$\begin{array}{ccc} & \text{Con}(X, M_\bullet)_{\text{rel } A} & \\ \cong \nearrow & & \nwarrow \cong \\ \text{Con}(X, Z_\bullet)_{\text{rel } A} & \xrightarrow{f_*} & \text{Con}(X, W_\bullet)_{\text{rel } A} \end{array}$$

$(i^Z)_*$ $(i^W)_*$

□

Corollary 9.10 *Let X be a topological space having the homotopy type of a CW-complex, and let Z_\bullet be a good simplicial space, then there exists a simplicial space \tilde{Z}_\bullet together with a simplicial map $Z_\bullet \rightarrow \tilde{Z}_\bullet$ such that the induced maps $\text{Con}(X, Z_\bullet) \rightarrow \text{Con}(X, \tilde{Z}_\bullet)$ and $[X, |Z_\bullet|] \rightarrow [X, |\tilde{Z}_\bullet|]$ both are bijections. Moreover, \tilde{Z}_\bullet satisfies the topological Kan condition.*

Proof: Let \mathcal{F} be the set of maps between simplicial spaces consisting of the family

$$\{\Lambda_\bullet^{n,k} \hookrightarrow \Delta_\bullet^n\}$$

and the family

$$\{D^m \times I \times \Lambda_\bullet^{n,k} \cup_{D^m \times \{0\} \times \Lambda_\bullet^{n,k}} D^m \times \Delta_\bullet^n \hookrightarrow D^m \times I \times \Delta_\bullet^n\} \quad .$$

We may perform Quillen's small object argument on the map $Z_\bullet \rightarrow *$ with respect to this set \mathcal{F} , see [DS95, proposition 7.17]. This produces a fibrant replacement \tilde{Z}_\bullet of Z_\bullet together with a simplicial map $Z_\bullet \rightarrow \tilde{Z}_\bullet$. $\tilde{Z}_\bullet \rightarrow *$ has the right lifting property with respect to all maps in \mathcal{F} . Hence \tilde{Z}_\bullet satisfies the topological Kan condition. The geometric realizations of Z_\bullet and \tilde{Z}_\bullet are obviously weakly equivalent. Moreover, the lemma 9.3 and theorem 9.6 ensures that $\text{Con}(-, Z_\bullet)$ and $\text{Con}(-, \tilde{Z}_\bullet)$ are in natural one-to-one correspondence.

□

References

- [BC03] M. Bullejos and A. M. Cegarra, *On the geometry of 2-categories and their classifying spaces*, *K-Theory* **29** (2003), no. 3, 211–229. MR MR2028502 (2004i:18009)
- [BC04] John C. Baez and Alissa S. Crans, *Higher-dimensional algebra. VI. Lie 2-algebras*, *Theory Appl. Categ.* **12** (2004), 492–538 (electronic). MR MR2068522 (2005m:17039)
- [BDR04] Nils A. Baas, Bjørn Ian Dundas, and John Rognes, *Two-vector bundles and forms of elliptic cohomology*, *Topology, geometry and quantum field theory*, London Math. Soc. Lecture Note Ser., vol. 308, Cambridge Univ. Press, Cambridge, 2004, pp. 18–45. MR MR2079370 (2005e:55007)
- [Bek04] Tibor Beke, *Higher Čech theory*, *K-Theory* **32** (2004), no. 4, 293–322. MR MR2112899 (2005i:18007)
- [Bén67] Jean Bénabou, *Introduction to bicategories*, Reports of the Midwest Category Seminar, Springer, Berlin, 1967, pp. 1–77. MR MR0220789 (36 #3841)
- [DE72] Eldon Dyer and S. Eilenberg, *An adjunction theorem for locally equiconnected spaces*, *Pacific J. Math.* **41** (1972), 669–685. MR MR0319143 (47 #7689)
- [DI04] Daniel Dugger and Daniel C. Isaksen, *Topological hypercovers and \mathbb{A}^1 -realizations*, *Math. Z.* **246** (2004), no. 4, 667–689. MR MR2045835 (2005d:55026)
- [DS95] W. G. Dwyer and J. Spaliński, *Homotopy theories and model categories*, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, pp. 73–126. MR MR1361887 (96h:55014)
- [Dus02] John W. Duskin, *Simplicial matrices and the nerves of weak n -categories. I. Nerves of bicategories*, *Theory Appl. Categ.* **9** (2001/02), 198–308 (electronic), CT2000 Conference (Como). MR MR1897816 (2003f:18005)
- [GJ99] Paul G. Goerss and John F. Jardine, *Simplicial homotopy theory*, Progress in Mathematics, vol. 174, Birkhäuser Verlag, Basel, 1999. MR 2001d:55012
- [Gra80] John W. Gray, *Closed categories, lax limits and homotopy limits*, *J. Pure Appl. Algebra* **19** (1980), 127–158. MR MR593251 (82f:18007a)

- [Hea86] Philip R. Heath, *A pullback theorem for locally-equiconnected spaces*, Manuscripta Math. **55** (1986), no. 2, 233–237. MR MR833247 (87f:55005)
- [KV94] M. M. Kapranov and V. A. Voevodsky, *2-categories and Zamolodchikov tetrahedra equations*, Algebraic groups and their generalizations: quantum and infinite-dimensional methods (University Park, PA, 1991), Proc. Sympos. Pure Math., vol. 56, Amer. Math. Soc., Providence, RI, 1994, pp. 177–259. MR MR1278735 (95f:18011)
- [ML98] Saunders Mac Lane, *Categories for the working mathematician*, second ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998. MR 2001j:18001
- [MS00] Michael K. Murray and Daniel Stevenson, *Bundle gerbes: stable isomorphism and local theory*, J. London Math. Soc. (2) **62** (2000), no. 3, 925–937. MR MR1794295 (2001j:55019)
- [MW04] Ib Madsen and Michael Weiss, *The stable moduli space of Riemann surfaces: Mumford’s conjecture*, arXiv:math.AT/0212321, July 2004.
- [Seg74] Graeme Segal, *Categories and cohomology theories*, Topology **13** (1974), 293–312. MR 50 #5782
- [Spa91] Edwin H. Spanier, *Algebraic topology*, Springer-Verlag, New York, 1991, Corrected reprint of the 1966 original. MR 96a:55001
- [Str87] Ross Street, *The algebra of oriented simplexes*, J. Pure Appl. Algebra **49** (1987), no. 3, 283–335. MR MR920944 (89a:18019)
- [Str96] ———, *Categorical structures*, Handbook of algebra, Vol. 1, North-Holland, Amsterdam, 1996, pp. 529–577. MR MR1421811 (97j:18007)
- [Wei05] Michael Weiss, *What does the classifying space of a category classify ?*, Homology, Homotopy and Applications **7** (2005), 185–195, <http://intlpress.com/HHA>.
- [Whi34] Hassler Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. **36** (1934), no. 1, 63–89. MR MR1501735
- [WS06] James Wirth and Jim Stasheff, *Homotopy Transition Cocycles*, Preprint, arXiv:math.AT/0609220, September 2006.