Løysingsforslag Eksamen ST1101, juni 2011

Oppgåve 1

$$f(x) = cx^2 \qquad 0 \le x \le 1$$

a) $\int_0^1 cx^2 = c\left[\frac{x^3}{3}\right]_0^1 = c \cdot \frac{1}{3} = 1 \Rightarrow c = \underline{3}$

$$E[X] = \int_0^1 3x^3\,dx = \left[\frac{3}{4}x^4\right]_0^1 = \underline{\frac{3}{4}}$$

$$F_X(x) = \int_0^x 3t^2\,dt = \left[t^3\right]_0^x = x^3, \qquad 0 \le x \le 1$$

Oppgåve 2

a)

1. $A \subset S \qquad P(A) \ge 0$

2. $P(S) = 1$

3. $A \cap B = \emptyset \Rightarrow P(A \cup B) = P(A) + P(B)$

4. $A_i \cap A_j = \emptyset,\ \forall i \ne j \Rightarrow P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$.

b) $A \cap A^c = \emptyset, \quad A \cup A^c = S$.

$\Rightarrow 1 = P(A \cup A^c) = P(A) + P(A^c) \Rightarrow P(A^c) = 1 - P(A)$

Oppgåve 3

$$X \sim N(1, 2^2)$$

1. $P(X>2) = P\left(\frac{X-1}{2} > \frac{2-1}{2}\right) = 1 - \Phi\left(\frac{1}{2}\right) = 1 - 0.6915 = \underline{0.3085}$

2. $P(2 < X \leq 3) = P\left(\frac{2-1}{2} < \frac{X-1}{2} \leq \frac{3-1}{2}\right) = \Phi(1) - \Phi\left(\frac{1}{2}\right) = 0.8413 - 0.6915 = \underline{0.149}$

3. $P(X>2 \mid X>1) = \dfrac{P(X>2 \cap X>1)}{P(X>1)} = \dfrac{P(X>2)}{P(X>1)} = \dfrac{P(X>2)}{P\left(\frac{X-1}{2} > \frac{1-1}{2}\right)}$

$$= 2P(X>2) = 2 \cdot 0.3085 = \underline{0.617}$$

Oppgåve 4

$$Y = e^X$$

$$F_Y(y) = P(Y \leq y) = P(e^X \leq y) = P(X \leq \ln y)$$

$$= F_X(\ln y) =$$

$$f_Y(y) = f_X(\ln y) \cdot \frac{1}{y}, \quad y \geq 0$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma} e^{-\frac{1}{2\sigma^2}(\ln y - \mu)^2} \cdot \frac{1}{y}, \quad y \geq 0. \quad \text{o: lognormalfordelt.}$$

Oppgåve 6

$$P(\min\{X_1, \ldots, X_n\} \leq y) = 1 - P(X_1 \geq y \cap X_2 \geq y \cap \cdots \cap X_n \geq y)$$

$$1 - (1 - F_X(y))^n \qquad F_X(y) = y$$

$$\Rightarrow P(Y \leq y) = 1 - (1-y)^n = F_Y(y)$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = n(1-y)^{n-1} \qquad 0 \le y \le 1$$

$$P\left(Y \le \frac{1}{n+1}\right) = 1 - \left(1 - \frac{1}{n+1}\right)^n = 1 - \left(\frac{n}{n+1}\right)^n$$

Oppgåve 8

$$f(x,\lambda) = f(x|\lambda) \cdot f(\lambda)$$

$$= \frac{\mu^s}{\Gamma(s)} \lambda^{s-1} e^{-\mu\lambda} \cdot \frac{\lambda^x e^{-\lambda}}{x!}, \quad \lambda > 0, \quad x = 0, 1, 2, \ldots$$

$$f_X(x) = \int_0^\infty \frac{\mu^s}{\Gamma(s)} \lambda^{s-1} e^{-\mu\lambda} \frac{\lambda^x e^{-\lambda}}{x!} d\lambda$$

$$= \frac{\mu^s}{\Gamma(s)\, x!} \int_0^\infty \lambda^{s+x-1} \frac{e^{-\lambda(\mu+1)}}{x!} d\lambda, \quad x = 0, 1, 2, \ldots$$

$$\overset{t=\lambda(\mu+1)}{=} \frac{\mu^s}{\Gamma(s)\, x!} \int_0^\infty \frac{t^{s+x-1}}{(\mu+1)^{x+s}} e^{-t} dt$$

$$= \frac{\mu^s}{\Gamma(s)\, x!} \frac{\Gamma(s+x)}{(\mu+1)^{x+s}} \overset{s \text{ heiltal}}{=} \frac{\mu^s}{(s-1)!\, x!} \frac{(x+s-1)!}{(\mu+1)^s (\mu+1)^x}$$

$$= \frac{(x+s-1)!}{(s-1)!\, x!} \left(\frac{\mu}{\mu+1}\right)^s \left(1 - \frac{\mu}{\mu+1}\right)^x$$

$$= \binom{x+s-1}{x} \left(\frac{\mu}{\mu+1}\right)^s \left(1 - \frac{\mu}{\mu+1}\right)^x = \binom{x+s-1}{s-1} \left(\frac{\mu}{\mu+1}\right)^s \left(1 - \frac{\mu}{\mu+1}\right)^x$$

$x = 0, 1, 2, \ldots$

Har form som ei negativ binomisk fordeling

$P(X = x+s)$ der $X =$ talet på hendingar for A skjer s-te gong, $p = \frac{\mu}{\mu+1}$