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1a) $B = A \cup (B \cap A^c) \Rightarrow P(B) = P(A) + P(B \cap A^c)$ sidan

A og $B \cap A^c$ er disjunkte.

Vi får: $P(A) = P(B) - P(B \cap A^c) \leq P(B)$ sidan $P(B \cap A^c) \geq 0$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} \quad \text{sidan } A \subset B \Rightarrow A \cap B = A$$

b).

$$P(\text{alle gutter}) = \left(\tfrac{1}{2}\right)^4 = \frac{1}{16}$$

$$P(\text{alle gutter} \mid \text{minst en er gutt}) = \frac{P(\text{alle gutter})}{1 - P(\text{alle er jenter})} = \frac{\left(\frac{1}{2}\right)^4}{1 - \left(\frac{1}{2}\right)^4} = \frac{\frac{1}{16}}{\frac{15}{16}} = \underline{\frac{1}{15}}$$

$$P(\text{alle gutter} \mid \text{minst 3 er gutter}) = \frac{P(\text{alle gutter})}{P(3\text{ gutter}) + P(4\text{ gutter})} = \frac{\left(\frac{1}{2}\right)^4}{\binom{4}{3}\left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^4} = \frac{1}{5}$$

2a)

$P(T=k) = P(A^c \text{ skjer i dei } k-1 \text{ første forsøka og } A \text{ skjer i } k\text{-te forsøk})$

$$= P(\underbrace{A^c \cap A^c \cap \cdots \cap A^c}_{k-1} \cap A) \overset{\text{p.g.a uavh.}}{=} P(A^c)^{k-1} \cdot P(A) = \left(\frac{11}{12}\right)^{k-1} \cdot \frac{1}{12}$$

Dette er ei geometrisk fordeling.

$$P(T > k) = \sum_{x=k+1}^{\infty} \left(\frac{11}{12}\right)^{x-1} \cdot \frac{1}{12} = \frac{1}{12} \cdot \left(\frac{11}{12}\right)^k \cdot \frac{1}{1 - \frac{11}{12}} = \left(\frac{11}{12}\right)^k$$

$$P(T \leq k) = 1 - P(T > k) = 1 - \left(\frac{11}{12}\right)^k.$$

$$M_T'(t) = \frac{pe^t}{1-(1-p)e^t} + \frac{(1-p)e^t \cdot pe^t}{(1-(1-p)e^t)^2} = \frac{pe^t}{(1-(1-p)e^t)^2}$$

$$M_T'(0) = \frac{p}{p^2} = \frac{1}{p}$$

$$M_T''(t) = \frac{pe^t}{(1-(1-p)e^t)^2} + \frac{2(1-p)e^t \cdot pe^t}{(1-(1-p)e^t)^3}$$

$$M_T''(0) = \frac{p}{p^2} + \frac{2(1-p)\cdot p}{p^3} = \frac{p+2-2p}{p^2} = \frac{2-p}{p^2}$$

$$Var(T) = M_T''(0) - (M_T'(0))^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

Innsett $p = \frac{1}{12}$ gjev $E(T) = 12$ og $Var(T) = \frac{1-\frac{1}{12}}{(\frac{1}{12})^2} = 11 \cdot 12 = \underline{132}$

c) N er negativt binomisk fordelt med parameter $r = 15$ og $p = \frac{1}{12}$. N gjev talet på forsøk som må gjerast før hendinga skjer for 15. gong.

$$P(N = k) = \binom{k-1}{r-1}\left(\frac{1}{12}\right)^r \cdot \left(\frac{11}{12}\right)^{k-r}, \quad k = r, r+1, \ldots$$

$N = \sum_{i=1}^{15} T_i$ der T_i er geometrisk fordelt som gitt i a.

$\Rightarrow E(N) = 15 \cdot 12 = 180$ og $Var(N) = 15 \cdot 132 = \underline{1980}$

d) $N \approx N(180, (\sqrt{1980})^2)$

$$P(N > 220) = P(N \geq 221) = P\left(\frac{N-180}{\sqrt{1980}} \geq \frac{221-180}{\sqrt{1980}}\right)$$

$$= 1 - \Phi(0.92) = 1 - 0.821 = 0.179$$

$P(\text{Hans må spørje meir enn 20 bil}) = P(T > 20) = \left(\frac{11}{12}\right)^{20} = 0.175$

e)

$X \sim B(1095, \frac{1}{365})$ p.g.a. $\begin{cases} \text{uavh. forsøk} \\ \text{Reg same dato / ikkje same dato} \\ p \text{ er konstant} \end{cases}$

n er stor og p er liten $\Rightarrow$ $P(X=x) \approx P(Z=x)$ der Z er Poissonfordelt med parameter $\lambda = np = \frac{1095}{365} = 3$

$$P(X=0) \approx P(Z=0) = \frac{3^0 e^{-3}}{0!} = e^{-3}$$

3a)

La X vere bremselengda til ein bil som køyrer i 40 km/t

$X \sim N(8, (1.6)^2)$

$$P(X < 6) = P\left(\frac{X-8}{1.6} \le \frac{6-8}{1.6}\right) = \Phi\left(\frac{-2}{1.6}\right) = \Phi(-1.25) = 0.106$$

$$P(7 \le X \le 9) = P\left(\frac{7-8}{1.6} \le \frac{X-8}{1.6} \le \frac{9-8}{1.6}\right) = \Phi\left(\frac{1}{1.6}\right) - \Phi\left(-\frac{1}{1.6}\right)$$

$$= \Phi(0.625) - \Phi(-0.625) = 0.7339 - 0.2661 = \underline{0.464}$$

b)

$E[Y] = 16\theta = E[\bar{Y}]$

$16\hat{\theta}_e = 9.6 = \bar{y} \Rightarrow \hat{\theta}_e = \underline{0.6}$

$$P\left(-1.96 < \frac{\bar{Y} - 16\theta}{\frac{1.6}{\sqrt{8}}} < 1.96\right) = 0.95$$

$$\Leftrightarrow P\left(\bar{Y} - 1.96 \cdot \frac{1.6}{\sqrt{8}} < 16\theta < \bar{Y} + 1.96 \cdot \frac{1.6}{\sqrt{8}}\right) = 0.95$$

$$\Leftrightarrow P\left(\frac{\bar{Y}}{16} - 1.96 \cdot \frac{0.1}{\sqrt{8}} < \theta < \frac{\bar{Y}}{16} + 1.96 \cdot \frac{0.1}{\sqrt{8}}\right) = 0.95$$

$\Rightarrow$ 95% konfidensintervallet blir:

$$(0.6 - 0.069\ ,\ 0.6 + 0.069) = (0.531,\ 0.669)$$

c)

$$L(\theta) = \prod_{i=1}^{6} \frac{1}{\sqrt{2\pi}} \frac{1}{16\sigma} e^{-\frac{1}{2}\left(\frac{y_i - 16\theta}{16\sigma}\right)^2} \cdot \prod_{i=7}^{8} \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{25\sigma} e^{-\frac{1}{2}\left(\frac{y_i - 25\theta}{25\sigma}\right)^2}$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^8 \cdot \left(\frac{1}{16\sigma}\right)^6 \cdot \left(\frac{1}{25\sigma}\right)^2 e^{-\frac{1}{2}\sum_{i=1}^{6}\left(\frac{y_i - 16\theta}{16\sigma}\right)^2 - \frac{1}{2}\sum_{i=7}^{8}\left(\frac{y_i - 25\theta}{25\sigma}\right)^2}$$

$$\ln L(\theta) = -8\ln\sqrt{2\pi} - 6\ln(16\sigma) - 2\ln(25\sigma) - \frac{1}{2}\left(\frac{1}{16\sigma}\right)^2 \sum_{i=1}^{6}(y_i - 16\theta)^2 - \frac{1}{2}\left(\frac{1}{25\sigma}\right)^2 \sum_{i=7}^{8}(y_i - 25\theta)^2$$

$$\frac{\partial \ln L}{\partial \theta} = \left(\frac{1}{16\sigma}\right)^2 \sum_{i=1}^{6}(y_i - 16\theta)\cdot 16 + \left(\frac{1}{25\sigma}\right)^2 \sum_{i=7}^{8}(y_i - 25\theta)25 = 0$$

$$\Longleftrightarrow \frac{1}{16}\sum_{i=1}^{6}(y_i - 16\theta) + \frac{1}{25}\sum_{i=7}^{8}(y_i - 25\theta) = 0$$

$$\Longleftrightarrow \frac{1}{16}\sum_{i=1}^{6} y_i - 6\theta + \frac{1}{25}\sum_{i=7}^{8} y_i - 2\theta = 0$$

$$\Rightarrow \theta_e = \frac{\frac{1}{16}\sum_{i=1}^{6} y_i + \frac{1}{25}\sum_{i=7}^{8} y_i}{8} \quad \text{og} \quad \hat{\theta} = \frac{\frac{1}{16}\sum_{i=1}^{6} Y_i + \frac{1}{25}\sum_{i=7}^{8} Y_i}{8}$$

$$E[\hat{\theta}] = \frac{\frac{1}{16}\cdot 6 \cdot 16\theta + \frac{1}{25}\cdot 2 \cdot 25\theta}{8} = \frac{8\theta}{8} = \theta$$

$$Var[\hat{\theta}] = \frac{\frac{1}{16^2}\cdot 6 \cdot 16^2 \cdot \sigma^2 + \frac{1}{25^2}\cdot 2 \cdot 25^2 \sigma^2}{64} = \underline{\frac{\sigma^2}{8}}$$