

FINAL PENSUM

Introduction

- Signals and functions;
- Pure tones
- Subject of Fourier analysis
- Periodic signals;
- Reduction to period 2π ;
- Orthogonality
- Fourier expansion of periodic signals;
- Finite-dimensional picture
- Signals with finite energy
- Norm and inner product
- Properties of norm and inner product:
 - Linearity and antilinearity
 - homogeneity of the norm
 - triangle inequality
 - Schwartz inequality
- Best approximation by a finite linear combination of exponential functions
- Orthogonal projection

Fourier series

- Best approximation by sums of exponential functions in $L^2(-\pi, \pi)$ and orthogonal projection (repetition) ;
- Bessel inequality;
- Expression for the error of approximation through Fourier coefficients;
- Decay of coefficients for smooth periodic functions;
- Formulation of the task: Parseval identity, Fourier expansion;
- Pattern of the proof:

- reduction the problem to the problem of approximation;
- approximation of “good” functions;
- approximation of arbitrary function by a “good” one
- approximation of “good” functions:
 - expression for partial sums;
 - Dirichlet kernel;
 - Properties of the Dirichlet kernel
 - proof of possibility of approximation of good functions
- Fact: approximation of arbitrary function from $L^2(-\pi, \pi)$ by “good” ones;
- Expansion of functions from $L^2(-\pi, \pi)$ into Fourier series. Parseval identity;
- Some examples;
- Properties of coefficients of Fourier series;
- Space $L^1(-\pi, \pi)$. Its relation to $L^2(-\pi, \pi)$.
- Fourier coefficients of functions from $L^1(-\pi, \pi)$;
- Riemann-Lebesgue lemma.
- Pointwise convergence: local nature;
- Convergence of expansion of step function.
- Convergence at the point of jump;

Signals and filters

- Spectra of a signal;
- Octave, harmonic scale etc.
- Problem of filtering, properties of filters:
 - linearity
 - invariance
 - continuity

Convolution and its applications.

- Convolution of periodic functions;
- Convolution of periodic functions (repeat);
- Convolution and Fourier coefficients, transfer function;
- Integral version of the triangle inequality;
- Smoothing property of convolution. Proof of the fact about approximation

Discrete Fourier transform.

- Setting of the “discrete” problem
- Remind: trapezoid method
- Formulas for (approximate) Fourier coefficients
- Linear algebra setting. Discrete Fourier transform appears.
- Remind: roots of unity, discrete orthogonality relation.
- Inverse matrix, inverse transform
- Interpretation in terms of interpolation problem
- Elementary properties of discrete Fourier transform
- Periodization of a discrete signal, discrete (circular) convolution.
- Circular convolution and discrete Fourier transform
- Discrete Parseval identity
- Idea of uncertainty principle
- Support of signal
- Formulation of uncertainty principle
- Proof of the uncertainty principle
- Idea of the Fast Fourier transform
- Complete algorithm

Fourier transform in $L^1(-\infty, \infty)$.

- The space $L^1(-\infty, \infty)$.

- Definition of Fourier transform
- Fourier transform of a function from $L^1(-\infty, \infty)$ is a bounded continuous function.
- Fourier transform of derivatives
- Riemann-Lebesgue lemma
- Elementary properties of the Fourier transform:
 - parity
 - Fourier transform of real and pure complex functions
- examples of Fourier transform:
 - $f(t) = 1$ if $|t| < a$; and $f(t) = 0$ if $|t| \geq a$;
 - $f(t) = e^{-a|t|}$.
 - $f(t) = e^{-at^2}$.
- Translation and modulation and their relation to Fourier transform
- Heisenberg relation

$$\int \hat{f}g = \int f\hat{g}.$$

- Formula for inversion of Fourier transform, assuming $f, \hat{f} \in L^1$.
- Idea of the proof
- Proof of the inversion formula:
 - auxiliary functions $\{g_n\}$, behavior of $\{g_n\}$ and $\{\hat{g}_n\}$;
 - Passing to the limit
- Integral in principal value, examples
- Fact (*without proof*): inversion formula for a piecewise continuous, piecewise smooth function.
- Sinc function - analog of the Dirichlet kernel

AROUND CONVOLUTION

- Repeat: definition of convolution
- *Example*: Convolution of the step function with itself

- Hints from example:
 - increasing of support
 - regularization property
 - a little bit about polynomial splines
- regularisation by convolution
- support of the convolution of two finitely supported functions:
 - recall: definition of support
 - definition of the sum of two sets
 - support of convolution contains in the sum of supports
 - *without proof* Titchmarsh's theorem about convolution
- convolution of two functions from $L^1(\mathbb{R})$ belongs to $L^1(\mathbb{R})$
- Fourier transform of convolution of two functions from $L^1(\mathbb{R})$
- relation between convolution and decay
- Spaces $L^p(\mathbb{R})$, $1 \leq p \leq \infty$.
- discrete analogs: spaces l^p , $1 \leq p \leq \infty$
- Conjugate exponents, Hölder inequality
- Minkovsky inequality: triangle inequality for convolution (*without proof, but with comments*)
- convolution of functions from L^p and L^1

Spaces \mathcal{S} and \mathcal{D}

- definition of the space \mathcal{S}
- Fourier transform of a function from \mathcal{S} is in \mathcal{S}
- example of finitely supported infinitely smooth function
- definition of the space \mathcal{D}
- regularization by mean of convolution
- approximation of unity (*regularizing sequence* in terms of our textbook)
- fuctions from \mathcal{D} are dense in $L^1(\mathbb{R})$

- each function from \mathcal{S} is the Fourier transform of some function from \mathcal{S}
- convolution (and product) of two functions from \mathcal{S} belongs to \mathcal{S}
- summary: list of basic properties of functions from \mathcal{S} .

Fourier transform in $L^2(-\infty, \infty)$.

- recall definition and properties of $L^2(-\infty, \infty)$:
 - norm
 - inner product
 - Schwartz inequality
- **Fact:** $L^2(-\infty, \infty)$ is complete
- Task: to construct the Fourier transform for functions from $L^2(-\infty, \infty)$
 - why one cannot define the Fourier transform by an explicit formula
 - relation between $L^2(-\infty, \infty)$ and $L^1(-\infty, \infty)$
 - plan of construction
- Parseval equality for functions from \mathcal{S} .
- Limit procedure and definition of Fourier transform for functions from $L^2(-\infty, \infty)$ relation for inner product for functions from \mathcal{S} , say
- Parseval relation for functions from \mathcal{S} , say
- Prolongation of these relations to functions from $L^2(-\infty, \infty)$
- Schwartz inequality, applied to product of two functions from $L^2(-\infty, \infty)$
- inverse Fourier transform; $L^2(-\infty, \infty)$ is invariant with respect to Fourier transform
- List of properties of Fourier transform
- Fourier transform of decaying functions
- Integrals in mean value. Understanding Fourier transform in mean value sense
- important Fourier transforms:
 - Let $a \in \mathbb{C}$ and $\Re a > 0$. Then

$$\mathcal{F} : \frac{1}{a + 2i\pi t} \mapsto e^{a\xi} u(-\xi), \quad \mathcal{F} : \frac{1}{a - 2i\pi t} \mapsto e^{-a\xi} u(\xi),$$

–

$$\mathcal{F} : \frac{\sin t}{t} \mapsto \pi \chi_{[-1/2\pi, 1/2\pi]}(\xi)$$

- effective duration of a signal
- formulation of uncertainty principle (Heisenberg inequality), relation to quantum mechanics
- proof of the Heisenberg inequality
- Gaussian is an extremal function for the Heisenberg inequality
- other versions of the uncertainty principle
- Convolution of functions from $L^2(-\infty, \infty)$.

– definition of convolution

– $f, g \in L^2(\mathbb{R}) \Rightarrow f * g$ is bounded, continuous and vanishes at ∞

–

$$\mathcal{F}(fg) = \hat{f} * \hat{g}, \quad \mathcal{F}(\hat{f}\hat{g}) = f * g$$

–

$$f * g \in L^1(\mathcal{R}) \Rightarrow \mathcal{F}(f * g) = \hat{f}\hat{g}$$

Linear systems

- general setting of the problem
- reduction to multiplication after Fourier transform
- case when input/output belong to \mathcal{S} , polynomial case
- versions of solvability for polynomial case
- general transmitting systems
- causality, condition for causality
- examples
- general case: functions are not from \mathcal{S} : list of obstacles
- explain what causality is
- the simplest example of casual filter
- digression to Matte 4: calculation of integrals and condition of causality
- Examples of filters:

- low pass filter
- integrators and differentiators
- RC and RCL filters – this for self reading. please recall this, you had it from Kreiszig !!

Distributions

- Examples when functions are not enough
 - delta function
 - derivative of Heaviside function
 - Fourier transform of smth unbounded
 - etc.
- Linear functionals on test functions:
 - common feature of situation when functions are not enough
 - Spaces of test functions: \mathcal{D} and \mathcal{S}
 - Notion of linear functional: linearity + continuity
 - continuity on \mathcal{D}
 - continuity on \mathcal{S}
 - notation: \mathcal{D}' and \mathcal{S}'
- Examples
 - delta function
 - constant
 - polynomial
 - sum of delta functions, Dirac comb
 - regular distribution
 - example of distribution from \mathcal{D}' but not from \mathcal{S}'
- elementary properties of distributions:
 - support
 - odd and even distributions
 - periodic distributions
- what one can do with distributions
 - sum of distributions
 - periodization of distributions

- derivative of distribution
- examples of derivatives
- Fourier transform of distributions from \mathcal{S}'
- examples of Fourier transforms
- **BUT:** one cannot multiply distributions
- still product of a distribution against a function
- Derivatives
 - repeat definition of derivative of distribution
 - derivatives of functions with jumps and edges
 - again about splines
 - distribution $PV(1/x)$
 - distribution $PV(1/x^2)$
- Convergence
 - definition of convergence
 - distribution as a limit of regular ones
 - series of distributions
 - examples: Dirac comb, sum of exponential functions
 - slowly increasing sequences and the corresponding trigonometrical series
- Fourier transform of distributions
 - definition of Fourier transform
 - multiplication of tempered distribution by a polynomial and exponential function

Fourier transform of distributions(*cont.*)

- recall: definition of Fourier transform of a distribution
- multiplication of a distribution by a polynomial or exponential function
- derivative and shift of the Fourier transform of tempered distribution
- Fourier transform of the delta function
- Fourier transform of identical one
- Poisson summation formula

- Poisson summation formula for more general class of function
- multiplication of a tempered distribution by a function:
 - impossible, generally speaking
 - case when it is possible: extra smoothness and growth limitations
 - another case when it is still possible: relation between supports (recall definition of support first)
 - general approach to multiplication of a distribution by a function (*though I would prefer to avoid this operation if possible*)

Distributions with compact support

- Recall definition of support of a distribution; Distributions with compact support
- *Corollary:* if two test functions coincide in a neighborhood of the support of a distribution, the value of the distribution on the functions is the same
- Fourier transform of a distribution with compact support
- Distribution version of Paley-Wiener theorem
- *Corollary:* A version of uncertainty principle
- *Digression:* What they call by "Paley-Wiener theorem" has very little in common with what Paley-Wiener did
- "Real" Paley-Wiener theorem: formulation
- "Real" Paley-Wiener theorem: idea of the proof
- Notion of bandlimited functions

Convolution and Fourier transform

- Convolution of $\phi \in \mathcal{S}'$ and $f \in \mathcal{S}$.
- Fourier transform of such convolution

Distributions with compact support

- Recall definition of support of a distribution; Distributions with compact support
- *Corollary:* if two test functions coincide in a neighborhood of the support of a distribution, the value of the distribution on the functions is the same

- Fourier transform of a distribution with compact support
- Distribution version of Paley-Wiener theorem
- *Corollary:* A version of uncertainty principle
- *Digression:* What they call by "Paley-Wiener theorem" has very little in common with what Paley-Wiener did
- "Real" Paley-Wiener theorem: formulation
- "Real" Paley-Wiener theorem: idea of the proof
- Notion of bandlimited functions

Convolution and Fourier transform

- Convolution of $\phi \in \mathcal{S}'$ and $f \in \mathcal{S}$.
- Fourier transform of such convolution

Lecture 22 November 8

Bandlimited functions

- (the classical) Paley-Wiener theorem
 - Recall the formulation
 - Proof of sufficiency
 - Idea of proof of necessity
 - Bandlimited functions
- Sampling of bandlimited functions
 - setting of the problem
 - Whitaekker-Kotel'nikov-Shannon formula
 - WKS formula fails generally speaking for signals with unbounded energy
 - Nyquist frequency
 - WKS formula via Lagrange interpolation formula

Short time Fourier transform (SFT)

- Advantages and disadvantages of the Fourier transform
- Idea of Short time Fourier transform (Windowed Fourier transform)

- Definition of Short time Fourier transform
- Notion of time-frequency plane
- Statement: energy conservation of the SFT
- Statement: inversion formula for SFT
- (Two) expressions for SFT through the Fourier transform
- Proof of energy conservation formula
- proof of inversion formula