Proof for the existence of θ_6

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Existence of θ_6

Theorem (Lin–Wang–Xu)

h_6^2 survives to the E_∞-page in the Adams spectral sequence.

By Browder's theorem

Corollary

There exist framed manifolds with Kervaire invariant 1 in dimension 126.

Together with theorems of Browder 1969, Mahowald–Tangora 1967, Barratt–Jones–Mahowald 1984, and Hill–Hopkins–Ravenel 2016

Corollary

Framed manifolds with Kervaire invariant one exist in and only in dimensions 2, 6, 14, 30, 62, 126.

- dim 30, explicit manifold known by J.Jones 1978
- dim 62 and 126, no explicit manifold known

Ingredients of our proof

- Lin's program
  - noncommutative Gröbner bases for Steenrod algebra
  - secondary Steenrod algebra
  - propagation of differentials and extensions
- Techniques from $H\mathbb{F}_2$-synthetic/filtered spectra
  - Generalized Leibniz Rule
  - Generalized Mahowald Trick
- Adhoc arguments near stem 126
  - Barratt–Jones–Mahowald's inductive approach
  - upgraded by Burklund–Xu in the context of $H\mathbb{F}_2$-synthetic/filtered spectra:
    - θ_6 exists $\Leftrightarrow \lambda\eta\theta_5^2 = 0$
  - detailed analysis around stem 126 to prove $\lambda\eta\theta_5^2 = 0$

Rigidity theorems in $\mathrm{SH}(\mathbb{C})$

- $\mathrm{SH}(\mathbb{C})$: motivic stable homotopy category over $\mathbb{C}$
- bigraded spheres $S^{n,w}$
- $\tau\colon \widehat{S^{0,-1}} \to \widehat{S^{0,0}}$
- (Voevodsky): $\pi_{*,*}\mathrm{H}\mathbb{F}_p^{\mathrm{mot}} \cong \mathbb{F}_p[\tau]$, $|\tau| = (0,-1)$
- Betti realization:

$$\begin{aligned} \mathrm{SH}(\mathbb{C}) &\longrightarrow \mathrm{SH} \\ S^{n,w} &\longmapsto S^n \\ \mathrm{H}\mathbb{F}_p^{\mathrm{mot}} &\longmapsto \mathrm{H}\mathbb{F}_p \\ \tau &\longmapsto 1 \end{aligned}$$

- (Morel, Levine, Dugger–Isaksen, Hu–Kriz–Ormsby): motivic Adams and Adams–Novikov spectral sequence

Rigidity theorems in $\mathrm{SH}(\mathbb{C})$

- The motivic Adams–Novikov spectral sequence is Rigid!
- τ-bockstein spectral sequence

$$\text{mot ANSS} \xrightarrow{\tau^{-1}} \text{ANSS}$$

$$E_2 \cong \text{ANSS } E_2[\tau]$$

$$d_{2r+1}x = \tau^r y \longleftrightarrow d_{2r+1}x = y$$

- $\widehat{S^{0,0}}/\tau$: the cofiber of τ
- mot ANSS$(\widehat{S^{0,0}}/\tau)$ collapses at E_2

$$\pi_{*,*}\widehat{S^{0,0}}/\tau \cong \mathrm{Ext}^{*,*}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*)$$

- (Dugger–Isaksen, Gheorghe–Wang–Xu):

$$\mathrm{SH}_p^\wedge \xleftarrow[\text{generic fiber}]{\tau^{-1}} \widehat{S^{0,0}}\text{-Mod} \xrightarrow[\text{special fiber}]{mod\ \tau} \mathcal{D}(\mathrm{BP}_*\mathrm{BP}\text{-Comod})$$

Rigidity theorems in $\mathrm{Syn}_{\mathrm{H}\mathbb{F}_p}$

- $\mathrm{Syn}_{\mathrm{H}\mathbb{F}_p}$: stable homotopy category of synthetic $\mathrm{H}\mathbb{F}_p$-spectra
- bigraded spheres $S^{n,w}$
- λ: $S^{0,-1} \to S^{0,0}$
- $\pi_{*,*}\mathrm{H}\mathbb{F}_p^{\mathrm{syn}} \cong \mathbb{F}_p[\lambda]$, $|\lambda| = (0,-1)$

$$\mathrm{SH}_p^\wedge \xleftarrow[\text{generic fiber}]{\lambda^{-1}} \mathrm{Syn}_{\mathrm{H}\mathbb{F}_p} \xrightarrow[\text{special fiber}]{mod\ \lambda} \mathcal{D}(A_*\text{-Comod})$$

- (Pstragowski, Gheorghe–Isaksen–Krause–Ricka):

$$\mathrm{SH}(\mathbb{C})_{\mathrm{cell}}^\wedge \simeq \mathrm{Syn}_{\mathrm{BP}} \simeq \mathrm{Fil}_{\mathrm{BP}}$$

- (Burklund–Hahn–Senger):
  The synthetic Adams spectral sequence is Rigid!

Rigidity theorems in $\mathrm{Syn}_{\mathrm{H}\mathbb{F}_p}$

- The synthetic Adams spectral sequence is Rigid!
- λ-bockstein spectral sequence

$$\text{syn ASS} \xrightarrow{\lambda^{-1}} \text{ASS}$$

$$E_2 \cong \text{ASS } E_2[\lambda]$$

$$d_{r+1}x = \lambda^r y \longleftrightarrow d_{r+1}x = y$$

- $S^{0,0}/\lambda$: the cofiber of λ
- syn ASS$(S^{0,0}/\lambda)$ collapses at E_2

$$\pi_{*,*}S^{0,0}/\lambda \cong \mathrm{Ext}_{\mathrm{A}}^{*,*}(\mathbb{F}_p, \mathbb{F}_p)$$

- $\pi_{*,*}S^{0,0}/\lambda^n \longleftrightarrow$ Adams E_{n+1}-page

Examples

- Syn ASS $E_2 \cong$ ASS $E_2[\lambda]$
- Classical S^0: $d_2(h_4) = h_0 h_3^2$, $d_3(h_0 h_4) = h_0 d_0$
- $S^{0,0}$: $d_2(h_4) = \lambda h_0 h_3^2$, $d_3(h_0 h_4) = \lambda^2 h_0 d_0$
  - survive: $h_0 h_3^2$, $h_0 d_0$, $\lambda h_0 d_0$
- $S^{0,0}/\lambda$: $d_2(h_4) = \lambda h_0 h_3^2 = 0$, $d_3(h_0 h_4) = \lambda^2 h_0 d_0 = 0$
  - everything survives!
- $S^{0,0}/\lambda^2$: $d_2(h_4) = \lambda h_0 h_3^2$, $d_3(h_0 h_4) = \lambda^2 h_0 d_0 = 0$
  $d_2(\lambda h_4) = \lambda^2 h_0 h_3^2 = 0$
  - survive: $h_0 h_3^2$, $h_0 d_0$, $\lambda h_0 d_0$
    λh_4
- $\pi_{*,*} S^{0,0}/\lambda^n \longleftrightarrow$ Adams E_{n+1}-page

Differentials and Extensions

- $$S^{0,-n} \xrightarrow{\cdot\lambda^n} S^{0,0} \longrightarrow S^{0,0}/\lambda^n \xrightarrow[\text{total Adams diff}]{\delta_n} S^{1,-n}$$
- $n = 1, \qquad \mathrm{Ext}_A^{s,t} \cong \pi_{t-s,t} S^{0,0}/\lambda$
- $h_4 \in \mathrm{Ext}_A^{1,16} \cong \pi_{15,15+1} S^{0,0}/\lambda$
- h_4 doesn't lift to $S^{0,0}$
- $d_2(h_4) = \lambda h_0 h_3^2 \longleftrightarrow \delta_1(h_4) = h_0 h_3^2$
- $d_3(h_0 h_4) = \lambda^2 h_0 d_0 \longleftrightarrow \delta_1(h_0 h_4) = \lambda h_0 d_0$
- $\Rightarrow \qquad [h_0 h_3^2] \cdot [h_0] = [\lambda h_0 d_0]$ in $\pi_{14,14+4} S^{0,0}$
- Warning: in $\pi_{14,14+4} S^{0,0}$, the element $h_0 h_3^2$ detects two homotopy classes, differed by $\lambda[d_0]$!
  $\Rightarrow$ for the other choice of $[h_0 h_3^2]$, $\qquad [h_0 h_3^2] \cdot [h_0] = 0$.
- Define extensions on an Adams E_n-page,
  Translate differentials to extensions
- Generalized Leibniz Rule, Generalized Mahowald Trick

Notations

- For $x \in \mathrm{Ext}$, permanent cycle, we denote by
  - $\{x\}$: the set of all classes in π_* that are detected by x.
  - $[x]$: a specific or a general class in $\{x\}$, depending on the context.
- $\theta_5 = [h_5^2]$: any synthetic homotopy class in $\pi_{62,62+2}S^{0,0}$ detected by h_5^2 in the Adams E_2-page.
- Denote also by θ_5 its image in $\pi_{62,62+2}S^{0,0}/\lambda^r$ via the map $S^{0,0} \to S^{0,0}/\lambda^r$ for all $r \geqslant 1$.
- $\eta = [h_1] \in \pi_{1,1+1}S^{0,0}$.
- Fact: $2 \cdot \theta_5 = 0$ in $\pi_{62,62+2}S^{0,0}$ for every θ_5.
  - $\mathrm{Ext}_A^{0,63} = 0 \;\Rightarrow\; \pi_{62,62+2}S^{0,0}$ doesn't contain any λ-torsion.
  - Xu, Isaksen–Wang–Xu: $2 \cdot \theta_5 = 0$ in $\pi_{62}S^0$ for every θ_5.

Inductive Approach for θ_6

Theorem (Barratt–Jones–Mahowald, Burklund–Xu)

1. *The element h_6^2 survives to the E_{r+3}-page of the classical Adams spectral sequence if and only if for some θ_5,*

$$\lambda\eta\theta_5^2 = 0 \text{ in } \pi_{125,125+4}S^{0,0}/\lambda^{r+1}.$$

2. *In particular, h_6^2 is a permanent cycle in the classical Adams spectral sequence if and only if for some θ_5,*

$$\lambda\eta\theta_5^2 = 0 \text{ in } \pi_{125,125+4}S^{0,0}.$$

- In fact, the expression $\lambda\eta\theta_5^2$ is consistent for every choice of θ_5.

Ideas of the Proof

- Take any $\theta_5 \in \pi_{62,62+2}S^{0,0}$, and its extension $f : S^{62,64}/2 \to S^{0,0}$, where $S^{62,64}/2$ is the cofiber of $2 : S^{62,64} \to S^{62,64}$.
- Consider its quadratic construction $Sq(f) : (S^{62,64}/2)^{\wedge 2}_{hC_2} \to S^{0,0}$.

$$\begin{tikzcd}
\boxed{63,64} \arrow[d, dash, "2"] & \\
\boxed{62,64} \arrow[dr, "\theta_5"] & \\
& \boxed{0,0}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\boxed{127,128} \arrow[d, dash, "2"] & \\
\boxed{126,128} \arrow[d, dash, "\lambda^2\eta"] & \\
\boxed{124,128} \arrow[dr, "\theta_5^2"] & \\
& \boxed{0,0}
\end{tikzcd}$$

$$S^{62,64}/2 \xrightarrow{f} S^{0,0} \qquad\qquad (S^{62,64}/2)^{\wedge 2}_{hC_2} \xrightarrow{Sq(f)} S^{0,0}$$

Ideas of the Proof

$$
\begin{tikzcd}
\boxed{63,64} \arrow[d, dash, "2"] \arrow[dr, dashed, bend left, "h_6"] & \\
\boxed{62,64} \arrow[r, "\theta_5"'] & \boxed{0,0}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\boxed{127,128} \arrow[d, dash, "2"] \arrow[ddr, dashed, bend left, "h_7"] & \\
\boxed{126,128} \arrow[d, dash, "\lambda^2\eta"] \arrow[dr, dashed, bend left, "h_6^2"] & \\
\boxed{124,128} \arrow[r, "\theta_5^2"'] & \boxed{0,0}
\end{tikzcd}
$$

$$S^{62,64}/2 \xrightarrow{f} S^{0,0} \qquad\qquad (S^{62,64}/2)^{\wedge 2}_{hC_2} \xrightarrow{Sq(f)} S^{0,0}$$

$$\Longrightarrow \qquad \begin{aligned} S^{0,0}/\lambda &\xrightarrow{\delta_1} S^{1,-1} \\ h_6^2 &\longmapsto \lambda\eta\theta_5^2 \end{aligned}$$

- If $\eta\theta_5^2$ is detected by $\lambda^{n-5}T_n$ for some $T_n \in \mathrm{Ext}_A^{n,125+n}$, then there is a synthetic Adams differential $d_{n-2}(h_6^2) = \lambda^{n-3}T_n$.

Strategy for proving h_6^2 as a permanent cycle

- If $\eta\theta_5^2$ is detected by $\lambda^{n-5}T_n$ for some $T_n \in \mathrm{Ext}_A^{n,125+n}$, then there is a synthetic Adams differential $d_{n-2}(h_6^2) = \lambda^{n-3}T_n$.
- Goal: Show that $\lambda\eta\theta_5^2 = 0$ by Adams filtration (AF) estimation.
- Start with $\theta_5^2 \in \pi_{124,124+4}S^{0,0}$.
  - $\mathrm{Ext}_A^{i,125+i} = 0$ for $i \leqslant 2$
  - $\Rightarrow \pi_{124,124+4}S^{0,0}$ doesn't contain λ-torision.
- By inspection, $\mathrm{AF}(\theta_5^2) \geqslant 10$.
  - If $\mathrm{AF}(\theta_5^2) = 10$, then θ_5^2 is detected by $\lambda^6 h_0^2 x_{124,8}$.
- Next estimate $\eta\theta_5^2$.
  - $\mathrm{AF}(\lambda^3\eta[h_0^2 x_{124,8}]) \geqslant 14$.
  - If $\mathrm{AF}(\lambda^3\eta[h_0^2 x_{124,8}]) = 14$, then it is detected by $\lambda^6 h_1 h_4 x_{109,12}$.
  - If $\mathrm{AF}(\lambda^3\eta[h_0^2 x_{124,8}]) > 14$, then it is zero.

An η-extension

Proposition A

Exactly one of (1) and (2) is true:

(1) h_6^2 *survives to the* E_∞*-page.*

(2) $d_{12}(h_6^2) = h_1 h_4 x_{109,12} \neq 0$.

(2) *is true* $\Leftrightarrow$ (3), (4), (5) *are all true:*

(3) $d_6(x_{126,8,4} + x_{126,8}) = 0$.

(4) $\theta_5^2 = \lambda^6[h_0^2 x_{124,8}] \neq 0 \in \pi_{124,124+4} S^{0,0}$.

(5) $\lambda^3 \eta [h_0^2 x_{124,8}] = \lambda^6 [h_1 h_4 x_{109,12}] \in \pi_{125,125+8} S^{0,0}$.

- Choice of θ_5 in (4) doesn't matter,
- Choice of $[h_0^2 x_{124,8}]$ in (5) doesn't matter.

Proposition B

If (3) is true, then (5) is not true.

Proof of Proposition B

For the sake of contradiction, we assume (3) and (5) are both true.

Lemma 1

There exists $\alpha_1 = [x_{123,9} + h_0 x_{123,8}] \in \pi_{123,123+9} S^{0,0}/\lambda^9$,

$$\alpha_2 \in \pi_{124,124+13} S^{0,0}/\lambda^9, \ \alpha_3 \in \pi_{125,125+15} S^{0,0}/\lambda^9,$$

such that

1. $\lambda^3 \eta \cdot \alpha_1 = \lambda^3 [h_0^2 x_{124,8}] + \lambda^6 \alpha_2 \qquad \in \pi_{124,124+7} S^{0,0}/\lambda^9,$
   $\eta \cdot \alpha_2 = \lambda \cdot \alpha_3 \qquad \in \pi_{125,125+14} S^{0,0}/\lambda^9,$
2. $\lambda^3 \cdot \alpha_1 \cdot [h_0] = 0 \qquad \in \pi_{123,123+7} S^{0,0}/\lambda^9.$

Lemma 2

The synthetic Toda bracket

$$\langle \lambda^3 \alpha_1, [h_0], \eta \rangle \subset \pi_{125,125+7} S^{0,0}/\lambda^9$$

does not contain zero, and is detected by $\lambda^4 h_0^2 x_{125,9,2}$.

Proof of Proposition B

Lemma 3 (Corollary of Lemma 2)

$[\lambda^4 h_0^2 x_{125,9,2}] \cdot [h_0] = \lambda^6 [h_1 h_4 x_{109,12}] \neq 0 \in \pi_{125,125+8} S^{0,0}/\lambda^9.$

Lemma 4

$[\lambda^4 h_1 x_{121,7}] \cdot [h_2] = \lambda[\lambda^5 h_0^2 x_{125,9,2}] \in \pi_{125,125+5} S^{0,0}/\lambda^9.$

Sketch Proof of Proposition B:

- $$\begin{aligned} [\lambda^4 h_1 x_{121,7}] \cdot [h_2] \cdot [h_0] &= \lambda[\lambda^5 h_0^2 x_{125,9,2}] \cdot [h_0] \\ &= \lambda^8 [h_1 h_4 x_{109,12}] \neq 0 \qquad \in \pi_{125,125+6} S^{0,0}/\lambda^9. \end{aligned}$$
- $[\lambda^4 h_1 x_{121,7}] \cdot [h_0] \neq 0 \in \pi_{122,122+5} S^{0,0}/\lambda^9$, only possibilities:

  $$\lambda^6 [h_6 M d_0] \text{ in AF} = 11, \quad \lambda^7 [h_5 x_{91,11}] \text{ in AF} = 12.$$

  Both lift to $\pi_{*,*} S^{0,0}$.
- In both cases, $\lambda^4 [h_1 h_4 x_{109,12}]$ is a $\lambda[h_2]$-multiple in $\pi_{*,*} S^{0,0}$.
- $\Rightarrow$ in $\mathrm{ASS}(S^0/\nu)$, $h_1 h_4 x_{109,12}[0]$ must be killed by d_r for $r \leqslant 5$.

Proof of Proposition B

Lemma 3 (Corollary of Lemma 2)

$[\lambda^4 h_0^2 x_{125,9,2}] \cdot [h_0] = \lambda^6 [h_1 h_4 x_{109,12}] \neq 0 \in \pi_{125,125+8} S^{0,0}/\lambda^9.$

Lemma 4

$[\lambda^4 h_1 x_{121,7}] \cdot [h_2] = \lambda[\lambda^5 h_0^2 x_{125,9,2}] \in \pi_{125,125+5} S^{0,0}/\lambda^9.$

Sketch Proof of Proposition B:

- $$\begin{aligned} [\lambda^4 h_1 x_{121,7}] \cdot [h_2] \cdot [h_0] &= \lambda[\lambda^5 h_0^2 x_{125,9,2}] \cdot [h_0] \\ &= \lambda^8 [h_1 h_4 x_{109,12}] \neq 0 \qquad \in \pi_{125,125+6} S^{0,0}/\lambda^9. \end{aligned}$$
- $[\lambda^4 h_1 x_{121,7}] \cdot [h_0] \neq 0 \in \pi_{122,122+5} S^{0,0}/\lambda^9$, only possibilities:

  $$\lambda^6 [h_6 M d_0] \text{ in AF} = 11, \quad \lambda^7 [h_5 x_{91,11}] \text{ in AF} = 12.$$

  Both lift to $\pi_{*,*} S^{0,0}$.
- In both cases, $\lambda^4 [h_1 h_4 x_{109,12}]$ is a $\lambda[h_2]$-multiple in $\pi_{*,*} S^{0,0}$.
- $\Rightarrow$ in $\mathrm{ASS}(S^0/\nu)$, $h_1 h_4 x_{109,12}[0]$ must be killed by d_r for $r \leqslant 5$. Contradiction!

Sketch Proof of Lemma 1

Lemma 1

There exists $\alpha_1 = [x_{123,9} + h_0 x_{123,8}], \alpha_2, \alpha_3 \in \pi_{*,*} S^{0,0}/\lambda^9$, *such that*

1. $\lambda^3 \eta \cdot \alpha_1 = \lambda^3 [h_0^2 x_{124,8}] + \lambda^6 \alpha_2 \quad \in \pi_{124,124+7} S^{0,0}/\lambda^9,$
   $\eta \cdot \alpha_2 = \lambda \cdot \alpha_3 \quad \in \pi_{125,125+14} S^{0,0}/\lambda^9,$
2. $\lambda^3 \cdot \alpha_1 \cdot [h_0] = 0 \quad \in \pi_{123,123+7} S^{0,0}/\lambda^9.$

Facts: (1) $x_{123,9} + h_0 x_{123,8}$ survives to E_{12}, not killed by any differential.
(2) $d_2(x_{125,8}) = h_1(x_{123,9} + h_0 x_{123,8}) + h_0^2 x_{124,8}$.

- Let $\alpha_1 = [x_{123,9} + h_0 x_{123,8}] \in \pi_{123,123+9} S^{0,0}/\lambda^{11}$ and its images in $\pi_{123,123+9} S^{0,0}/\lambda^r$ for $1 \leqslant r \leqslant 10$.
- $\lambda \eta \alpha_1 + \lambda [h_0^2 x_{124,8}] \in \pi_{124,124+9} S^{0,0}/\lambda^9$ lies in AF $\geqslant 11$.
- $\lambda^3 \eta \alpha_1 + \lambda^3 [h_0^2 x_{124,8}] \in \pi_{124,124+7} S^{0,0}/\lambda^9$ has AF $\geqslant 13$ by inspection, and is detected by $\lambda^6 e_0 \Delta h_6 g$ if AF $= 13$.
- in Ext, $h_1 \cdot e_0 \Delta h_6 g = 0$.
- $\lambda^3 \cdot \alpha_1 \cdot [h_0] \in \pi_{123,123+7} S^{0,0}/\lambda^{11}$ has AF $\geqslant 17$ by inspection.

Sketch Proof of Lemmas 2 and 3

Lemma 2

The synthetic Toda bracket $\langle \lambda^3\alpha_1, [h_0], \eta\rangle \subset \pi_{125,125+7}S^{0,0}/\lambda^9$ *does not contain zero, and is detected by* $\lambda^4 h_0^2 x_{125,9,2}$.

- From Lemma 1 and (5): $\lambda^3\eta[h_0^2 x_{124,8}] = \lambda^6[h_1 h_4 x_{109,12}]$,

$$\begin{aligned}\eta\cdot\lambda^3\eta\alpha_1 &= \eta\cdot\lambda^3[h_0^2 x_{124,8}] + \eta\cdot\lambda^6\alpha_2\\ &= \lambda^6[h_1 h_4 x_{109,12}] + \lambda^7\alpha_3 \in \pi_{125,125+8}S^{0,0}/\lambda^9.\end{aligned}$$

$$\eta\cdot\lambda^3\eta\alpha_1 = \lambda^3\alpha_1\cdot\langle[h_0],\eta,[h_0]\rangle = \langle\lambda^3\alpha_1,[h_0],\eta\rangle\cdot[h_0].$$

- $\Rightarrow$ $\langle\lambda^3\alpha_1,[h_0],\eta\rangle$ lies in AF $\leqslant 12$, which is generated by

$$\begin{aligned}[h_0^2 x_{125,5}] &\text{ in AF} = 7,\\ \lambda^2[h_6(\Delta e_1 + C_0 + h_0^6 h_5^2)] &\text{ in AF} = 9,\\ [\lambda^4 h_0^2 x_{125,9,2}] &\text{ in AF} = 11.\end{aligned}$$

- $\Rightarrow$ Lemma 3:
$[\lambda^4 h_0^2 x_{125,9,2}]\cdot[h_0] = \lambda^6[h_1 h_4 x_{109,12}] \neq 0 \in \pi_{125,125+8}S^{0,0}/\lambda^9$.

Sketch Proof of Lemma 4

Lemma 4

$[\lambda^4 h_1 x_{121,7}] \cdot [h_2] = \lambda[\lambda^5 h_0^2 x_{125,9,2}] \in \pi_{125,125+5} S^{0,0}/\lambda^9.$

- Fact: $h_1 x_{121,7}$ survives to E_6, not killed by any differential.
  $\Rightarrow [\lambda^4 h_1 x_{121,7}] \neq 0 \qquad \in \pi_{122,122+4} S^{0,0}/\lambda^9.$

$$S^3 \xrightarrow{\nu} S^0 \xrightarrow{i} S^0/\nu \xrightarrow{q} S^4 \xrightarrow{\nu} S^1$$

$$\cdots \xrightarrow{\cdot h_2} \mathrm{Ext}_A^{*,*}(S^0) \xrightarrow{i_*} \mathrm{Ext}_A^{*,*}(S^0/\nu) \xrightarrow{q_*} \mathrm{Ext}_A^{*,*}(S^4) \xrightarrow{\cdot h_2} \cdots$$

$$\begin{tikzcd}
h_0^2 x_{125,9,2} \arrow[r, mapsto] & h_0^2 x_{125,9,2}[0] & \\
& \begin{array}{c} h_1 x_{121,7}[4] \\ +x_{126,8}[0] \\ +x_{126,8,2}[0] \end{array} \arrow[u, "d_3"] \arrow[r, mapsto] & h_1 x_{121,7}
\end{tikzcd}$$

- Generalized Mahowald Trick:
  $[h_1 x_{121,7}] \cdot [h_2] = \lambda^2 [h_0^2 x_{125,9,2}] \in \pi_{125,125+9} S^{0,0}/\lambda^3.$
- Lift via $S^{0,0}/\lambda^5 \to S^{0,0}/\lambda^3$, push via $\lambda^4 : \Sigma^{0,-4} S^{0,0}/\lambda^5 \to S^{0,0}/\lambda^9$

Hopf, Kervaire, and $\cdots$

- In the Adams spectral sequence, $\mathrm{Ext}_A^{1,*}(\mathbb{F}_2, \mathbb{F}_2)$ is generated by the classes h_j.
- (Adams): h_j survives $\Leftrightarrow$ $j \leqslant 3$.
- The tangent bundle over S^n is trivial $\Leftrightarrow$ $n = 1, 3, 7$.
- (Hill–Hopkins–Ravenel, Lin–Wang–Xu): h_j^2 survives $\Leftrightarrow$ $j \leqslant 6$.
- There exists a framed n-dim manifold with Kervaire invariant one $\Leftrightarrow$ $n = 2,\ 6,\ 14,\ 30,\ 62, 126$.
- Question: Explicit differentials on h_j^2 for $j \geqslant 7$?
- (Burklund–Xu): h_j^3 survives $\Leftrightarrow$ $j \leqslant 4$.
- Ongoing progress: interpretation in terms of framed manifolds.
- $h_j^4 = 0$. Question: What's next?

New Doomsday Conjecture

- in Ext, $Sq^0 : \mathrm{Ext}_{\mathrm{A}}^{s,t} \longrightarrow \mathrm{Ext}_{\mathrm{A}}^{s,2t}$,

$$Sq^0 h_j = h_{j+1}, \ Sq^0 h_j^2 = h_{j+1}^2, \ Sq^0 h_j^3 = h_{j+1}^3$$

- Sq^0-family: $x, \ Sq^0 x, \ Sq^0(Sq^0 x), \ \cdots$
- New Doomsday Conjecture: For any nonzero Sq^0-family, only finitely many classes survive.
    - $\mathrm{Ext}_{\mathrm{A}}^{1,*} \Leftrightarrow$ Hopf invariant problem,
    - $\mathrm{Ext}_{\mathrm{A}}^{2,*} \Leftrightarrow$ Kervaire invariant problem,
    - $\mathrm{Ext}_{\mathrm{A}}^{3,*}$: other than h_j^3, many cases remain

$$h_j^2 h_{j+k+1} + h_{j+1} h_{j+k}^2 = \langle h_j^2, \ h_0, \ h_{j+k}^2 \rangle.$$

- Uniform Doomsday Conjecture: For any nonzero Sq^0-family $\{a_j\}$, there exists a Sq^0-family $\{b_j\}$, $r \geqslant 2$, $c \in \mathrm{Ext}$, such that

$$d_r(a_j) = c \cdot b_j \neq 0, \ \text{for } j >> 0.$$

Thank you!