Dependent type theory and higher algebraic structures

Chaitanya Leena Subramaniam

IRIF, Université de Paris (Paris Diderot)

ACPMS seminar

17 Sep. 2021
Part of this talk is joint work with Peter LeFanu Lumsdaine.

This is part of a bigger project on the correspondence between algebraic structures in homotopy theory and dependently sorted theories.

For details, see PhD thesis (available at sites.google.com/view/chaitanyals).

Some of this was developed during a visit to Stockholm financed by the Fondation Sciences mathématiques de Paris.
Algebraic structures on sets

Many algebraic structures (such as monoids, groups, rings, modules, algebras etc.) have an underlying set (or family of sets).

A set can be seen as a 0-dimensional object (a collection of points).
“Cellular” algebraic structures

Certain algebraic structures don’t have an underlying set, but have an underlying “cellular” structure that has non-zero dimension.

For example, a small category has an underlying (directed) graph (with cells of dimension 0 and 1).
A coloured operad has an underlying collection (cells in dimension 0 (colours) and 1 (operations)).

An \textit{n-category} has an underlying \textit{n-graph} or \textit{n-globular set} (cells in dimension \(\leq n\)).
A simplicial set has an underlying semi-simplicial set (cells in all finite dimension).

Also cubical sets, globular sets, opetopic sets etc.
Algebraic structures up to homotopy

Notions of “cellularity” also appear in algebraic structures arising in homotopy theory.

- E.g. $A_\infty$-spaces are the “correct” notion of associative monoids in spaces.
- An ordinary monoid (with an underlying set) can be seen as a 0-truncated $A_\infty$-space.
Higher algebraic structures are the right notion of algebraic structures in homotopy theory.

Many higher algebraic structures (such as $A_\infty$-spaces and $E_k$-spaces) have underlying spaces.

In fact, every space is already an interesting example of a higher algebraic structure since it is an $\infty$-groupoid.
Grothendieck’s **homotopy hypothesis** is the idea that a space should be represented by a cellular algebraic structure (a cellular algebraic $\infty$-groupoid).

Its 0-dimensional structure consists of its points, its 1-dimensional structure are its paths between points, and so on in every finite dimension ($(n+1)$-paths between parallel $n$-paths).
More generally, we can conjecture that the theory of cellular algebraic structures “fits” well with that of higher (or homotopical) algebraic structures.

The goal of this talk is to partially justify this intuition, using the universal algebra and homotopy theory of dependently sorted (or typed) algebraic theories.
Universal algebra of cellular algebraic structures

The universal algebra of algebraic theories was described by Lawvere, Bénabou and Linton.

Multisorted algebraic theories correspond to those algebraic structures with underlying sets (or families of sets), namely 0-dimensional algebraic structures.

In this talk, we will see that dependently sorted/typed algebraic theories correspond to higher dimensional cellular algebraic structures.
Outline

We will

1. Define dependently sorted algebraic theories,
2. give a precise definition of “cellular algebraic structure”,
3. and describe how they interact with homotopy theory.
Dependent type theory vs. dependently sorted theories

A theory is a set of axioms expressed in a language.

\(^1\) (For experts: *structural rules only.*)
A theory is a set of axioms expressed in a language.

Dependent type theory (DTT) is a language:
it is Martin-Löf’s framework of dependent types.\(^1\)

\(^1\)(For experts: structural rules only.)
Dependent type theory vs. dependently sorted theories

A theory is a set of axioms expressed in a language.

Dependent type theory (DTT) is a language:
It is Martin-Löf’s framework of dependent types.\(^1\)

A dependently sorted (algebraic) theory is a theory:
It is a set of types, terms and equalities expressed in DTT.

\(^1\)(For experts: structural rules only.)
Rules of DTT

\[
\begin{align*}
\vdash \diamond \text{ctxt} & \quad \text{EMP} & \quad \Gamma \vdash A \text{ type} & \quad \text{EXT} \\
\vdash \Gamma, x : A \text{ ctxt} & \quad \text{EXT} \\
\vdash \Gamma, x : A, \Delta \text{ ctxt} & \quad \text{VAR} \\
\Gamma, x : A, \Delta \vdash x : A & \quad \text{VAR} \\
\vdash \Gamma, \Delta \vdash J & \quad \Gamma \vdash A \text{ type} & \quad \text{WEAK} \\
\Gamma, x : A, \Delta \vdash J & \quad \text{WEAK} \\
\vdash \Gamma, x : A, \Delta \vdash J & \quad \Gamma \vdash a : A & \quad \text{SUBST} \\
\Gamma, \Delta[a/x] \vdash J[a/x] & \quad \text{SUBST}
\end{align*}
\]

...not important for the rest of the talk.
Algebraic theories as dependently sorted theories

“Baby” theories don’t have type dependency — e.g. the theory of abelian groups:

\[ \vdash G \text{ type} \]
\[ \vdash 1 : G \]
\[ x : G \vdash x^{-1} : G \]
\[ x, y : G \vdash x \cdot y : G \]
\[ x, y, z : G \vdash (x \cdot y) \cdot z = x \cdot (y \cdot z) : G \]
\[ x, y : G \vdash x \cdot y = y \cdot x : G \]
\[ x : G \vdash x \cdot 1 = x : G \]
\[ x : G \vdash x \cdot x^{-1} = 1 : G \]

also of monoids, groups, rings, modules, algebras, bialgebras, Hopf algebras . . .
Example of a dependently sorted theory

E.g. of an “adult” theory — the theory of categories:

\[ \vdash \text{Ob type} \]

\[ x, y : \text{Ob} \vdash \text{Hom}(x, y) \text{ type} \]

\[ x : \text{Ob} \vdash 1_x : \text{Hom}(x, x) \]

\[ \ldots, f : \text{Hom}(x, y), g : \text{Hom}(y, z) \vdash g \circ f : \text{Hom}(x, z) \]

\[ x, y : \text{Ob}, f : \text{Hom}(x, y) \vdash f \circ 1_x = f = 1_y \circ f : \text{Hom}(x, y) \]

\[ \ldots \vdash (h \circ g) \circ f = h \circ (g \circ f) : \text{Hom}(x_1, x_4) \]

also of 2-categories, \( \omega \)-categories, reflexive graphs, semisimplicial sets, opetopic sets \ldots
Sneak peek at the big picture

Operations in a “baby” theory

An operation in a multisorted Lawvere theory takes a finite coproduct of points as input, and outputs a point.
Sneak peek at the big picture

Operations in an “adult” theory

Every operation in a dependently sorted theory takes a finite cell complex as input, and outputs a cell.

(This is related to Burroni-Leinster $T$-operads.)
An **algebraic theory** \( T \) consists of:

1. A set \( S \) of sorts,
2. a set \( F \) of \( S \)-sorted function symbols, each written
   \[
   x: A_1, \ldots, x_n: A_n \vdash g(x_1, \ldots, x_n) : A \quad (A_1, \ldots, A_n, A \in S).
   \]
3. A set \( E \) of equations between terms over \( F \), each written
   \[
   x:A_1, \ldots, x_n:A_n \vdash t = u : A.
   \]
A model (in Set) of $\mathbf{T}$ consists of:

1. A set $X_A$ for each sort $A \in \Sigma$,

2. a function $X_f : X_{A_1} \times \ldots \times X_{A_n} \to X_A$ for every function symbol $x : A_1, \ldots, x_n : A_n \vdash f : A$ in $\mathbf{F}$,

3. satisfying every equation in $\mathbf{E}$ (in the obvious sense).
Multisorted example

Let $C$ be a small category, with set of objects $\text{ob}(C)$. The theory of (covariant) $C$-presheaves has:

1. set of sorts $\text{ob}(C)$,

2. for each morphism $f : A \to B$ in $C$, a function symbol $x : A \vdash f(x) : B$.

3. equations $x : A \vdash g(f(x)) = h(x) : C$ for every commutative triangle $\begin{tikzcd}
A & B \\
& C
\arrow[Rightarrow]{r}{f} & \arrow[Rightarrow]{l}{g} \arrow[Rightarrow]{u}{h}
\end{tikzcd}$ in $C$. 
Multisorted example

The theory of \((\text{single-coloured, planar } \text{Set}-)\text{operads}\) has:

1. set of sorts \(\{O_n \mid n \in \mathbb{N}\}\),

2. function symbols

\[ \vdash 1 : O_1 \quad f_1 : O_{n_1}, \ldots, f_k : O_{n_k}, g : O_k \vdash c(g, f_1, \ldots, f_k) : O_{n_1 + \ldots + n_k} \]

3. equations

\[ g : O_k \vdash c(g, 1, \ldots, 1) = g : O_k \]

\[ \ldots \vdash \text{(associativity is too long to write)} \]
**S-sorted sets**

Let $S$ be a set. Then an **$S$-sorted set** is a disjoint union of sets indexed by $S$:

$$X = \bigsqcup_{s \in S} X_s.$$  

In other words, it is a function $X \to S$.

The category of $S$-sorted sets is the presheaf category $\hat{S} \simeq \text{Set}/S$. 
Lawvere–Bénabou’s observation

Let $T$ be a “baby” theory (i.e. with no type-dependency) with a set $S$ of sorts.

Then every $T$-model has an underlying $S$-sorted set.

E.g. the theory of ring-module pairs: every ring-module pair $(a, m)$ has an underlying $\{A, M\}$-sorted set (i.e. a pair of sets).

E.g. for $C \in \text{Cat}$, the theory of $C$-presheaves: every presheaf $C \to \text{Set}$ has an underlying $\text{ob}(C)$-sorted set.
Lawvere (1963) and Bénabou (1968) gave an algebraic description of what a “baby” theory (i.e. no type-dependency) is.

**Theorem (Lawvere, Bénabou)**

*For every $S$-sorted algebraic theory $T$, there is a unique identity-on-objects, finite-product-preserving functor $\text{Fin}_{S}^{\text{op}} \to C_T$. The category of $T$-models is the category of finite-product-preserving functors $C_T \to \text{Set}$ (and all natural transformations).*
Every identity-on-objects, finite-product-preserving functor 
\( \text{Fin}_S^{op} \to C \) gives rise to a unique \( S \)-sorted algebraic theory \( T_C \)

- whose operations are the elements of the hom-sets 
  \( C(A_1 \times \ldots \times A_n, A) \) for every \( A_1, \ldots, A_n, A \in S \),

- and whose equations are given by commutative triangles in \( C \).

\[
A_1 \times \ldots \times A_n \xrightarrow{(f_1, \ldots, f_m)} B_1 \times \ldots \times B_m
\]
Definition

The category $\text{Law}_S$ of $S$-sorted algebraic theories is the category of identity-on-objects, finite-product-preserving functors $\text{Fin}_S^{op} \to C$ (with morphisms all commuting triangles in $\text{Cat}$ as below).

\[
\begin{array}{ccc}
\text{Fin}_S^{op} & \longrightarrow & C \\
\downarrow & & \downarrow \\
C & \longrightarrow & C'
\end{array}
\]
Linton’s “recognition theorem”

Theorem (Linton (1965))

A baby theory (a.k.a. multisorted algebraic theory) is exactly the data of

1. a set $S$ of “types” or “sorts”
2. and a finitary monad on $\widehat{S} = \text{Set}_S$.

Namely, we have an equivalence of categories

$$\text{Law}_S \simeq \text{FinMnd}(\widehat{S}).$$

Recall: A finitary monad is one whose underlying endofunctor preserves filtered colimits.
Goal: A “nice” definition of dependently sorted theory

We want to give an algebraic description à la Lawvere–Bénabou of “adult” dependently sorted theories.

This description should (obviously) strictly generalise Lawvere–Bénabou’s.
Syntactic dependently sorted algebraic theories

First question
What should a syntactic dependently sorted algebraic theory be?

Many approaches

- Cartmell’s generalised algebraic theories [Car78].
- Makkai’s logic with dependent sorts [Mak95].
- Fiore’s $\Sigma_n$-models with substitution [Fio08].
- Palmgren’s DFOL signatures [Pal16].
- Others (Aczel, Belo, QIITs ... )
Our syntactic definition is strictly less general than each of the above...
Our syntactic definition is strictly less general than each of the above.

... but it has a very good generalisation of the algebraic descriptions of Lawvere–Bénabou–Linton,
Our syntactic definition is strictly less general than each of the above. . .

. . . but it has a very good generalisation of the algebraic descriptions of Lawvere–Bénabou–Linton,

. . . as well as a good definition of models in spaces.
“Recognition theorem” for dependently typed theories

Theorem (LS, LeFanu Lumsdaine)

A \textit{dependently typed theory} is the data of

1. a \textit{locally finite direct (lfd) category} $\mathbf{C}$ of dependent types/sorts
2. and a \textit{finitary monad} on $\widehat{\mathbf{C}}$.

We have equivalences of categories

$$\text{CxlCat}_\mathbf{C} \simeq \text{Law}_\mathbf{C} \simeq \text{FinMnd}(\widehat{\mathbf{C}}).$$
Examples of lfd categories

1. Any set $S$ (seen as a discrete category).
Examples of lfd categories

1. Any set $S$ (seen as a discrete category).
2. The ordinal $\omega$ (seen as a totally ordered poset).
Examples of lfd categories

1. Any set $S$ (seen as a discrete category).

2. The ordinal $\omega$ (seen as a totally ordered poset).

3. The category $\mathbb{G}_1 = \{0 \Rightarrow 1\}$ with two objects and two parallel non-identity arrows.
Examples of lfd categories

1. Any set $S$ (seen as a discrete category).

2. The ordinal $\omega$ (seen as a totally ordered poset).

3. The category $\mathbb{G}_1 = \{0 \Rightarrow 1\}$ with two objects and two parallel non-identity arrows.

4. The category of globes $\mathbb{G}$ [Lei04, Def. 1.4.5].
Examples of lfd categories

1. Any set $S$ (seen as a discrete category).

2. The ordinal $\omega$ (seen as a totally ordered poset).

3. The category $\mathbb{G}_1 = \{0 \rightrightarrows 1\}$ with two objects and two parallel non-identity arrows.

4. The category of *globes* $\mathbb{G}$ [Lei04, Def. 1.4.5].

5. The category $\text{elTr}_p$ of *planar elementary trees* [Koc11, 2.4.4] or *planar corollas*. 
Examples of lfd categories

1. Any set $S$ (seen as a discrete category).

2. The ordinal $\omega$ (seen as a totally ordered poset).

3. The category $G_1 = \{0 \Rightarrow 1\}$ with two objects and two parallel non-identity arrows.

4. The category of *globes* $G$ [Lei04, Def. 1.4.5].

5. The category $\text{elTr}_p$ of *planar elementary trees* [Koc11, 2.4.4] or *planar corollas*.

6. The category $\mathcal{O}$ of *opetopes*.
7. Every Reedy category $R$ has a wide subcategory $R'$ that is direct. In many (if not most) well-known examples, $R'$ is also locally finite, such as:
7. Every Reedy category $R$ has a wide subcategory $R'$ that is direct. In many (if not most) well-known examples, $R'$ is also locally finite, such as:

7.1 $R = \Delta$, the simplex category ($\Delta'$ is called the semi-simplex category),
7. Every Reedy category $R$ has a wide subcategory $R'$ that is direct. In many (if not most) well-known examples, $R'$ is also locally finite, such as:

7.1 $R = \Delta$, the simplex category ($\Delta'$ is called the semi-simplex category),

7.2 $R = \Omega_p$, the planar dendroidal category [MT10, Def. 2.2.1],

($\Omega'_p$ is called the category of planar semi-dendrices),
7. Every Reedy category $R$ has a wide subcategory $R'$ that is direct. In many (if not most) well-known examples, $R'$ is also locally finite, such as:

7.1 $R = \Delta$, the simplex category ($\Delta'$ is called the semi-simplex category),

7.2 $R = \Omega_p$, the planar dendroidal category [MT10, Def. 2.2.1],

($\Omega'_p$ is called the category of planar semi-dendrices),

7.3 $R = \Theta$, Joyal’s cell category [Joy97],

where in each case $R'$ is the wide subcategory of monomorphisms.
Examples of dependently typed theories

- Every multisorted Lawvere theory.
Examples of dependently typed theories

- Every multisorted Lawvere theory.
- The identity monads on $\hat{G}_1$, $\hat{G}$, $\hat{O}$, $\hat{\Delta}'$. 
Examples of dependently typed theories

- Every multisorted Lawvere theory.
- The identity monads on $\widehat{G_1}$, $\widehat{G}$, $\widehat{O}$, $\widehat{\Delta'}$.
- The free-category monad on $\widehat{G_1}$. 
Examples of dependently typed theories

- Every multisorted Lawvere theory.
- The identity monads on $\hat{G}_1$, $\hat{G}$, $\hat{O}$, $\hat{\Delta}'$.
- The free-category monad on $\hat{G}_1$.
- The free-strict-$\omega$-category monad on $\hat{G}$. 
Examples of dependently typed theories

- Every multisorted Lawvere theory.
- The identity monads on $\hat{G}_1$, $\hat{G}$, $\hat{O}$, $\hat{\Delta}'$.
- The free-category monad on $\hat{G}_1$.
- The free-strict-$\omega$-category monad on $\hat{G}$.
- Every free-weak-$\omega$-category monad on $\hat{G}$.
Examples of dependently typed theories

- Every multisorted Lawvere theory.
- The identity monads on $\hat{G}_1$, $\hat{G}$, $\hat{O}$, $\hat{\Delta'}$.
- The free-category monad on $\hat{G}_1$.
- The free-strict-$\omega$-category monad on $\hat{G}$.
- Every free-weak-$\omega$-category monad on $\hat{G}$.
- For $T: \hat{C} \to \hat{C}$ a finitary cartesian monad, every $T$-operad (à la Burroni-Leinster).
Examples of dependently typed theories

- Every multisorted Lawvere theory.
- The identity monads on $\hat{G}_1$, $\hat{G}$, $\hat{O}$, $\hat{\Delta}'$.
- The free-category monad on $\hat{G}$.
- The free-strict-$\omega$-category monad on $\hat{G}$.
- Every free-weak-$\omega$-category monad on $\hat{G}$.
- For $T: \hat{C} \to \hat{C}$ a finitary cartesian monad, every $T$-operad (à la Burroni-Leinster).
- The free-planar-coloured-operad monad on $\hat{\text{elTr}}_p$. 
Examples of dependently typed theories

- Every multisorted Lawvere theory.
- The identity monads on $\widehat{G}_1, \widehat{G}, \widehat{O}, \widehat{\Delta'}$.
- The free-category monad on $\widehat{G}_1$.
- The free-strict-$\omega$-category monad on $\widehat{G}$.
- Every free-weak-$\omega$-category monad on $\widehat{G}$.
- For $T : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}$ a finitary cartesian monad, every $T$-operad (à la Burroni-Leinster).
- The free-planar-coloured-operad monad on $\widehat{\text{elTr}}_p$.
- The free-simplicial-set monad on $\widehat{\Delta'}$.
- And many more...
Categorical interlude
Let $\mathcal{C}$ be a small category.
Let $\mathcal{C}$ be a small category.

$\text{Fin}_\mathcal{C}$ is the category of finitely presentable presheaves on $\mathcal{C}$. Write the dense inclusion as $E : \text{Fin}_\mathcal{C} \hookrightarrow \hat{\mathcal{C}}$. 
Let $\mathcal{C}$ be a small category.

$\text{Fin}_{\mathcal{C}}$ is the category of finitely presentable presheaves on $\mathcal{C}$. Write the dense inclusion as $E : \text{Fin}_{\mathcal{C}} \hookrightarrow \hat{\mathcal{C}}$.

Recall that $\text{Fin}_{\mathcal{C}}$ is the finite-colimit completion of $\mathcal{C}$. When $\mathcal{C}$ is a set, $\text{Fin}_{\mathcal{C}}$ is the also the finite-coproduct completion of $\mathcal{C}$. 
Cartesian collections

The presheaf category

\[ \text{Coll}_C \overset{\text{def}}{=} [\text{Fin}_C \times C^\text{op}, \text{Set}] \]

is called the category of \textit{cartesian} \textit{C}-collections.

(Intuition: \( F \in \text{Coll}_C \) should be thought of as a \textit{term signature} — for each context \( \Gamma \in \text{Fin}_C \) and each sort \( s \in C \), \( F(s, \Gamma) \) is the set of “operations” with input \( \Gamma \) and output sort \( s \).)
Composition of cartesian collections

\[ \mathbf{C} \text{-collections can be composed via substitution:} \]

\[ G \circ F(i, \Gamma) \overset{\text{def}}{=} \int_{\Theta \in \text{Fin}_C} G(i, \Theta) \times \widehat{\mathbf{C}}(\Theta, F(-, \Gamma)). \]

\((\text{Coll}_{\mathbf{C}}, \circ, E)\) is a \textbf{monoidal category}, where \(E : \text{Fin}_C \leftrightarrow \widehat{\mathbf{C}}.\)
The functor \( \text{Lan}_E(-) : \text{Coll}_C \to \left[ \hat{\mathcal{C}}, \hat{\mathcal{C}} \right] \) of left Kan extension along \( E : \text{Fin}_C \to \hat{\mathcal{C}} \) is (1) fully faithful and (2) monoidal.

\[
\begin{array}{c}
\text{Fin}_C \xrightarrow{F} \hat{\mathcal{C}} \\
\hat{\mathcal{C}} \xrightarrow{\text{Lan}_E F} \hat{\mathcal{C}}
\end{array}
\]

(1)

\[
\text{Lan}_E (F \circ G) \cong \text{Lan}_E F \circ \text{Lan}_E G ; \quad \text{Lan}_E E \cong \text{id}
\]

(2)
**Consequence**

\[
\text{Lan}_E^- : \text{Mon}(\text{Coll}_C, \circ, E) \leftrightarrow \text{Mnd}(\widehat{C})
\]

The category of monoids in \( \text{Coll}_C \) is a full subcategory of the category of monads on \( \widehat{C} \). It is none other than the category of finitary monads on \( \widehat{C} \).
Consequence

\[ \text{Lan}_E : \text{Mon(Coll}_\mathcal{C}, \circ, E) \leftrightarrow \text{Mnd}(\hat{\mathcal{C}}) \]

The category of monoids in \( \text{Coll}_\mathcal{C} \) is a full subcategory of the category of monads on \( \hat{\mathcal{C}} \). It is none other than the category of finitary monads on \( \hat{\mathcal{C}} \).

Remarks

- We have only used that \( \mathcal{C} \) is a small category.
- \( \text{Mon(Coll}_\mathcal{C}) \) is also known as the category of monads with arities (Weber) or Lawvere theories with arities (Melliès) for the arities \( E : \text{Fin}_\mathcal{C} \leftrightarrow \hat{\mathcal{C}} \).
Dependently sorted algebraic theories
Syntactic example

The theory of small categories has:

1. signature of dependent sorts/dependent types

\[ \vdash O \quad x, y : O \vdash M(x, y) \]

2. function symbols are

\[ \vdash x : O \vdash 1(x) : M(x, x) \quad \ldots, f : M(x, y), g : M(y, z) \vdash g \circ f : M(x, z) \]

3. equations are

\[ \ldots \vdash 1(y) \circ f = f : M(x, y) \quad \ldots \vdash f \circ 1(x) = f : M(x, y) \]

\[ \ldots \vdash h \circ (g \circ f) = (h \circ g) \circ f : M(x_1, x_4). \]
A simpler example is the theory of *directed graphs*, which has:

1. signature of dependent sorts/dependent types

$$\vdash O \quad x, y : O \vdash M(x, y)$$

2. no function symbols
3. and no equations.

A model (in $\mathbf{Set}$) of this “theory” is the data of a set $X_O$, and for every $(x, y) \in X_O \times X_O$, a set $X_M(x, y)$. That is, a model is a diagram $X_M \Rightarrow X_O$ in $\mathbf{Set}$. 
Similarly the theory of 2-\textit{graphs} (2-globular sets) has a signature of dependent sorts

\[ \vdash O \quad x, y : O \vdash M(x, y) \]
\[ \{x, y : O\}, f, g : M(x, y) \vdash 2M(f, g) \]

and no function symbols or equations.

A model is a diagram in \textit{Set}

\[
\begin{array}{ccc}
X_{2M} & \xrightarrow{s} & X_M \\
& \xrightarrow{t} & \\
X_M & \xrightarrow{s} & X_0
\end{array}
\]

such that \( ss = st \) and \( ts = tt \).
Dependent type signatures and cellularity

Definition

A category is **locally finite** if all of its slice categories are finite.

Lemma

*If $C$ is a locally finite category and $X : C^{op} \to \text{Set}$ is a presheaf on $C$, then its category of elements $C/X$ is locally finite.*
Every small category $C$ has a relation on its set of objects $\text{ob}(C)$.

\[ c < d \iff \text{there exists a non-identity morphism } c \to d \]

**Definition**

A small category $C$ is **direct** if the relation $<$ is *well-founded*, i.e. there are no infinite descending chains $\ldots c_2 < c_1 < c_0$.

**Lemma**

*If $X : C^{\text{op}} \to \text{Set}$ is a presheaf on a direct category $C$, then its category of elements $C/X$ is direct.*
Definition

A type signature is a locally finite direct (lfed) category.
Let $\mathcal{C}$ be a lfd category.

- For every $c$ in $\mathcal{C}$, let $\mathcal{C}_{/c}$ be the full subcategory of the slice category $\mathcal{C}_{/c}$ obtained by removing the identity morphism $\text{id} : c \to c$.

- The **boundary** $\partial c \in \hat{\mathcal{C}}$ of $c$ is the sub-representable presheaf $\partial c \hookrightarrow c$ that is the colimit of the diagram $\mathcal{C}_{/c} \subset \mathcal{C}_{/c} \to \mathcal{C} \hookrightarrow \hat{\mathcal{C}}$.

- We define the set of **boundary inclusions** of $\mathcal{C}$ to be the following set of maps in $\hat{\mathcal{C}}$.

$$ I_\mathcal{C} \overset{\text{def}}{=} \{ \delta_c : \partial c \hookrightarrow c \mid c \in \mathcal{C} \} $$
Note that:

- The Yoneda embedding factors as $C \hookrightarrow \text{Fin}_C \hookrightarrow \hat{C}$. 
Note that:

- The Yoneda embedding factors as $C \hookrightarrow \text{Fin}_C \hookrightarrow \hat{C}$.

- The boundary inclusions $\partial \c c \hookrightarrow c$ are finitely presentable (since $C/c$ is finite).
Note that:

- The Yoneda embedding factors as $\mathcal{C} \hookrightarrow \text{Fin}_\mathcal{C} \hookrightarrow \hat{\mathcal{C}}$.
- The boundary inclusions $\partial c \hookrightarrow c$ are finitely presentable (since $\mathcal{C}/c$ is finite).
- Every finite cell complex in $\hat{\mathcal{C}}$ is finitely presentable.
Note that:

- The Yoneda embedding factors as $C \hookrightarrow \text{Fin}_C \hookrightarrow \widehat{C}$.
- The boundary inclusions $\partial c \hookrightarrow c$ are finitely presentable (since $C/c$ is finite).
- Every finite cell complex in $\widehat{C}$ is finitely presentable.
- Every $X \in \text{Fin}_C$ can be written as a finite cell complex.
Note that:

- The Yoneda embedding factors as $\mathcal{C} \hookrightarrow \text{Fin}_\mathcal{C} \hookrightarrow \hat{\mathcal{C}}$.
- The boundary inclusions $\partial c \hookrightarrow c$ are finitely presentable (since $\mathcal{C}/c$ is finite).
- Every finite cell complex in $\hat{\mathcal{C}}$ is finitely presentable.
- Every $X \in \text{Fin}_\mathcal{C}$ can be written as a finite cell complex.
- Every finite cell complex is of finite dimension (all cells are of dimension $< n$ for some $n$).
Definition

The category $\text{Cell}_C$ of finite $C$-sorted cell contexts has a graded set of objects $\text{ob}(\text{Cell}_C) \overset{\text{def}}{=} \bigsqcup_{n \in \mathbb{N}} (\text{Cell}_C)_n$ that are finite $I_C$-cell complexes $\emptyset \rightarrow \ldots X_n$ inductively defined as:

- $(\text{Cell}_C)_0$ consists only of the empty presheaf $\emptyset$,
- for every $\emptyset \rightarrow \ldots X_n$ in $(\text{Cell}_C)_n$, $c \in C$ and every span $c \xleftarrow{\delta_c} \partial c \xrightarrow{\gamma} X_n$ in $\widehat{C}$, we make a choice of pushout square

$$
\begin{array}{ccc}
\partial c & \xrightarrow{\gamma} & X_n \\
\delta_c \downarrow & & \downarrow p_{n+1} \\
\gamma.c & \xrightarrow{} & X_{n+1}
\end{array}
$$

giving $\emptyset \rightarrow \ldots \rightarrow X_n \xrightarrow{p_{n+1}} X_{n+1}$ in $(\text{Cell}_C)_{n+1}$.

The morphisms of $\text{Cell}_C$ are defined by

$\text{Cell}_C(\emptyset \rightarrow \ldots X, \emptyset \rightarrow \ldots Y) \overset{\text{def}}{=} \widehat{C}(X, Y)$. 
Contextual categories

Contextual categories are an algebraic gadget introduced by Cartmell, and there is a dependently sorted syntactic theory (a “generalised algebraic theory”) associated to each contextual category. We write $\text{CxlCat}$ for the category of contextual categories.
Contextual categories are an algebraic gadget introduced by Cartmell, and there is a dependently sorted syntactic theory (a “generalised algebraic theory”) associated to each contextual category. We write $\text{CxlCat}$ for the category of contextual categories.

**Theorem (LS, LeFanu Lumsdaine)**

$\text{Fin}_C$ is equivalent to $\text{Cell}_C$, and $\text{Cell}^{op}_C$ is a contextual category. Moreover it is the free contextual category on $C$, and we write it $\text{Cx}(C)$. 

Definition (**C-contextual category**)

Let $f : \text{Cx}(\mathcal{C}) \to D$ be a morphism in $\text{CxlCat}$, and let

$$\text{Cx}(\mathcal{C}) \xrightarrow{j_f^{op}} \Theta_f^{op} \hookrightarrow D$$

be the (identity-on-objects, fully faithful) factorisation of its underlying functor. Then $f$ is a **C-contextual category** if for every morphism $g : \text{Cx}(\mathcal{C}) \to D'$ in $\text{CxlCat}$ and every triangle $h j_f^{op} = g$ (in $\text{Cat}$), there exists a unique morphism $\tilde{h} : D \to D'$ in $\text{CxlCat}$ making the following diagram commute.

$$\text{Cx}(\mathcal{C}) \xrightarrow{j_f^{op}} \Theta_f^{op} \hookrightarrow D$$

$g$ $\downarrow h$ $\exists! \tilde{h}$

$D'$
The category $\mathbf{CxlCat}_C$ of $\mathbf{C}$-contextual categories is the full subcategory of the coslice category $\mathbf{CxlCat}_{\mathbf{C}x(\mathbf{C})/}$ consisting of the $\mathbf{C}$-contextual categories.

Theorem (LS, LeFanu Lumsdaine)

*The categories of $\mathbf{C}$-contextual categories and finitary monads on $\hat{\mathbf{C}}$ are equivalent.*

$$
\mathbf{CxlCat}_C \simeq \mathbf{FinMnd}(\hat{\mathbf{C}})
$$
Models of dependently sorted algebraic theories
Recall that a dependently typed algebraic theory $\mathbf{T}$ is the data of

1. a lfd category $\mathbf{C}$

2. and a finitary monad $T$ on $\mathbf{C}$.

**Definition**

The category $\mathbf{T}$-$\text{Mod}$ of $\mathbf{T}$-models is the category $T$-$\text{Alg}$ of algebras of the monad $T$.

Thus $\mathbf{T}$-$\text{Mod}$ is a locally finitely presentable (l.f.p.) category.
Morita equivalence with essentially algebraic theories

Theorem (LS)

Every l.f.p. category is the category of models of some dependently sorted algebraic theory over some lfd category $C$. 
Let $C$ be a lfd category, and let $T$ be a $C$-sorted algebraic theory. By the previous recognition theorems, there is a unique morphism of contextual categories $C_x(C) \to C_T$ corresponding to $T$.

Recall that we have an equivalence $\text{Fin}_{C}^{op} \simeq C_x(C)$.
Theorem (LS)

The category $T$-Mod is equivalent to the category of presheaves (and all natural transformations) $C_T \to \text{Set}$ such that the composite $\text{Fin}^\text{op}_C \to C_T \to \text{Set}$

1. takes the initial object $\emptyset \in \text{Fin}_C$ to the terminal set,
2. and takes every pushout square in $\text{Fin}_C$ of the form

$$
\begin{array}{ccc}
\partial c & \xrightarrow{\gamma} & X_n \\
\delta_c & \downarrow \gamma & \downarrow p_{n+1} \\
C & \xrightarrow{\gamma.c} & X_{n+1}
\end{array}
$$

...to a pullback square in $\text{Set}$.

Remark: when $C$ is a set, this exactly says that $\text{Fin}^\text{op}_C \to C_T \to \text{Set}$ preserves finite products.
Homotopy models

Definition

A homotopy-model of $\mathbf{T}$ is a functor $C_\mathbf{T} \to S$ to the $\infty$-category of spaces, that

1. takes the initial object $\emptyset \in \text{Fin}_C$ to the terminal space,
2. and takes every pushout square in $\text{Fin}_C$ of the form

$$
\begin{array}{ccc}
\partial c & \xrightarrow{\gamma} & X_n \\
\delta_c & \uparrow & \downarrow p_{n+1} \\
c & \xrightarrow{\gamma.c} & X_{n+1}
\end{array}
$$

...to a pullback square in $S$.

This gives an $\infty$-category of homotopy-models of $\mathbf{T}$. 
Theorem (LS)

For every dependently sorted algebraic theory $T$, there is a Quillen model category (a left Bousfield localisation of simplicial presheaves) that models the $\infty$-category of homotopy-models of $T$.

In some good cases of $T$, we have obtained a rigidification theorem generalising one due to Badzioch for ordinary algebraic theories.

I conjecture that this is true in general (for all dependently sorted algebraic theories). (Work-in-progress with S. Henry.)
J. W. Cartmell.

*Generalised algebraic theories and contextual categories.*

Marcelo Fiore.

Second-order and dependently-sorted abstract syntax.

André Joyal.

Disks, duality and $\theta$-categories.

Joachim Kock.

Polynomial functors and trees.
Tom Leinster.  
Higher Operads, Higher Categories.  

Michael Makkai.  
First Order Logic with Dependent Sorts, with Applications to Category Theory.  

Ieke Moerdijk and Bertrand Toën.  
Trees as operads.  

Erik Palmgren.
Categories with families, FOLDS and logic enriched type theory.