The Amplituhedron and Positive Geometries

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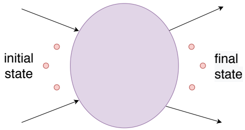
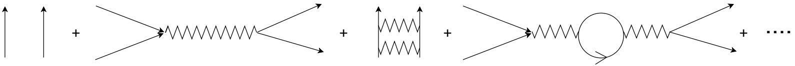
Scattering amplitudes in quantum field theory

- When particles interact, different outcomes are possible. The probability of a given outcome is called the scattering amplitude $\mathcal{A}(\mathrm{in}, \mathrm{out})$.



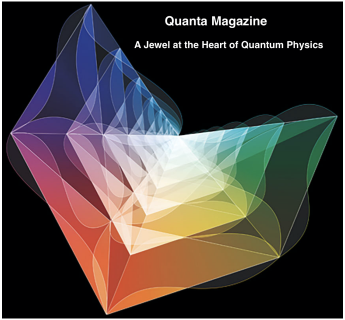
- Feynman diagrams are used for for explicit computations: $\int \mathcal{D}\phi e^{iS[\phi]}$



- Each Feynman diagram is the sum of exponentially many terms that contribute as a summand to $\mathcal{A}$.

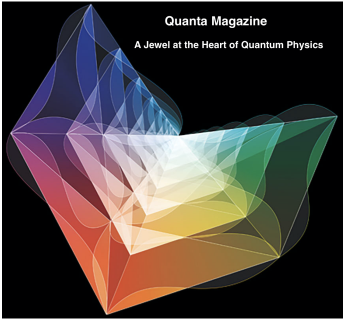
The Amplituhedron

- Novel “on-shell methods" developed to calculate scattering amplitudes such as the Britto-Cachazo-Feng-Witten (BCFW) recursive method
- Introduced by Nima Arkani-Hamed and Jaroslav Trnka in 2013
- A geometric basis of BCFW recursion is given in terms of a collection of cells in the positive Grassmannian
- These cells glue together and form the **Amplituhedron** $\mathcal{A}_{n,k,m}$.



Properties of amplituhedron

- The **volume** is scattering amplitudes for $\mathcal{N} = 4$ Super Yang-Mills.
- **The canonical form** $\Omega(\mathcal{A}_{n,k,m})$: a unique top-degree rational form whose poles are all simple and are exactly along the boundary of $\mathcal{A}_{n,k,m}$.
- The BCFW recurrence for computing scattering amplitudes is reformulated as giving a **triangulation** of the $m = 4$ amplituhedron.
- **Question:** How to find the canonical form and triangulations of $\mathcal{A}_{n,k,m}$?

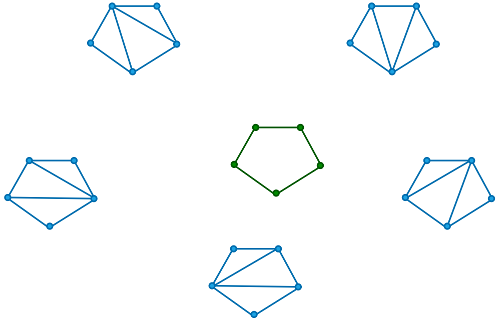


Outline

- Examples of amplituhedra and their triangulations
  - Triangulations of polytopes
  - Cyclic polytopes
  - Positive Grassmannians
- The Amplituhedron
  - Combinatorial properties
  - Generalized triangulations
- Positive geometries
  - Canonical forms
  - Further examples
- Summary and further directions

Subdivisions of polytopes

- A **subdivision** of a polytope is a finite union of polytopes s.t. every two polytopes are either disjoint or intersect by a common proper face.



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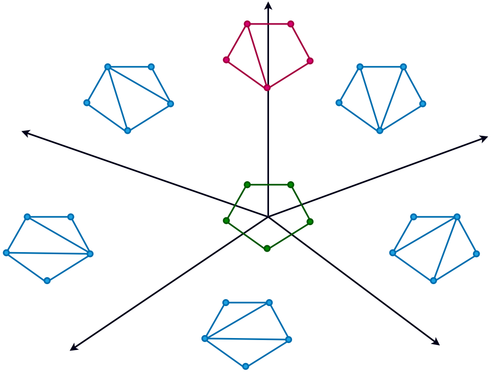


- **The secondary fan of** P**:** A fan whose faces correspond to the regular subdivisions of P. Gelfand-Kapranov-Zelevinski, 1990

Example 1: The cyclic polytope

- **The cyclic polytope**: the convex hull of any n distinct points on

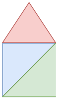
  the moment curve $\{x(t) = (t, t^2, \dots, t^d) : \ t \in \mathbb{R}\} \subset \mathbb{R}^d$.

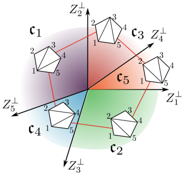
- The combinatorial type of $C(n, d)$ is independent of the choice of the points on the moment curve. Edelman-Reiner '96, Rambau '97

$$C(n, d) = \mathrm{conv}\{x(t_1), \dots, x(t_n)\} \ \text{ for } \ t_1 < \cdots < t_n$$

- A **triangulation** of a polytope P is a finite union of simplices s.t. every two simplices are either disjoint or intersect by a common proper face.
- **Secondary fan of P:** It is a polyhedral fan whose faces correspond to the regular subdivisions of P. Gelfand-Kapranov-Zelevinskii, '90

  Triangulations of $P \Longleftrightarrow$ Maximal dimensional cones of the secondary fan $\mathcal{F}_P$





Example 2: The positive Grassmannian

- **Grassmannian** $\mathrm{Gr}(k, n)$: The set of k-dim subspaces in $\mathbb{C}^n$ (or $\mathbb{R}^n$).
- Each element of $\mathrm{Gr}(k, n)$ is represented by a full-rank $k \times n$ matrix X.

  $\mathrm{Gr}(k, n)$ is $\mathrm{Mat}(k, n)/\sim$ (up to GL_k-action.)

$$X = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \in \mathrm{Gr}(2, 3) \quad \text{with} \quad P_{12} = 1, P_{13} = 3, P_{23} = 1$$

- **Plücker coordinate** P_I for $I \in \binom{[n]}{k}$: The minor of X on the columns I.
- **Positive Grassmannian** $\mathrm{Gr}_+(k, n)$: the set of elements X of real Grassmannian $\mathrm{Gr}(k, n)$ where $P_I(X) \geq 0$ for all $I \in \binom{[n]}{k}$.
- Based on seminal work on positivity by Lusztig 1994 for $(G/P)_{\geq 0}$, as well as Fomin and Zelevinsky 1999, Postnikov 2006, and Rietsch 2007.
- $\mathrm{Gr}_+(k, n)$ plays the role of the **simplex** in amplituhedron theory.

Question

How to **decompose** $\mathrm{Gr}_+(k, n)$ into some topological cells?

Positroid decomposition of $\mathrm{Gr}_+(k, n)$

- Matroid stratification of $\mathrm{Gr}(k, n)$ introduced by Gelfand-Goresky-MacPherson-Serganova in 1987
- Its restriction to $\mathrm{Gr}_+(k, n)$ studied by Postnikov 2006
- Given a collection $M \subset \binom{[n]}{k}$ the **positroid cell** of M is:

$$S_M = \{X \in \mathrm{Gr}_+(k, n) : \; P_I(X) > 0 \text{ if and only if } I \in M\}.$$

Theorem (Postnikov 2006, Postnikov-Speyer-Williams 2009)

Every non-empty S_M is a topological cell homeomorphic to an open ball. The positroid cells glue together to form a CW complex

$$\mathrm{Gr}_+(k, n) = \bigsqcup_M S_M.$$

The Amplituhedron $\mathcal{A}_{n,k,m}$

- Introduced by Arkani-Hamed and Trnka in 2013
- $\mathcal{A}_{n,k,m}$ is the image of the positive Grassmannian under a linear map.
- Let n, k, m be integers with $k + m \leq n$ and m even.
- Let Z be an $n \times (k + m)$ with positive maximal minors.

$$\pi : \mathrm{Gr}_{+}(k, n) \to \mathrm{Gr}(k, k + m) \quad \text{with} \quad C \mapsto Y = C.Z$$

- The image of π is called the amplituhedron

$$\mathcal{A}_{n,k,m} := \pi(\mathrm{Gr}_{+}(k, n)) \subset \mathrm{Gr}(k, k + m)$$

- The definition of the amplituhedron depends on the choice of Z. Its combinatorial/geometric properties are independent of the choice of Z (similar to cyclic polytopes).

The properties of $\mathcal{A}_{n,k,m}$

- The amplituhedron

$$\mathcal{A}_{n,k,m} := \pi(\mathrm{Gr}_+(k,n)) \subset \mathrm{Gr}(k,k+m)$$

  is a semialgebraic subset of $\mathrm{Gr}(k,k+m)$
- It is of full-dimension km.
- $\mathcal{A}_{n,k,m}$ is the image of $\mathrm{Gr}_+(k,n)$ which decomposes into positroid cells.

$$\mathrm{Gr}_+(k,n) = \bigsqcup_M S_M.$$

- **Idea:** To use the images of positroid cells to triangulate $\mathcal{A}_{n,k,m}$.
- $\pi(\mathrm{Gr}_+(k,n))$ is of full-dimension, but $\pi(S_M)$ might have lower dimension.

Question

Find collections $\mathcal{C} = \{S_M\}$ of positroid cells whose **images** triangulate $\mathcal{A}_{n,k,m}$.

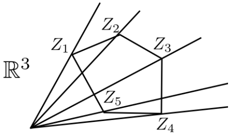
Triangulations of $\mathcal{A}_{n,k,m}$

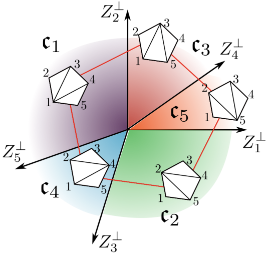
- **Triangulation of** $\mathcal{A}_{n,k,m}$: A collection of positroid cells $\mathcal{C} = \{S_M\}$ s.t.
  - (1) $\pi(S_M)$ are of full-dimensional km, they are pairwise disjoint, and cover a dense subset of the amplituhedron.
  - (2) The map π is injective on each S_M in $\mathcal{C}$.
- **How do the boundary of these cells overlap?**
- **Good triangulations:**
  - (3) if the intersection $\overline{\pi_+(S_M)} \cap \overline{\pi_+(S_{M'})}$ of the images of two distinct cells has codimension one, then $\overline{\pi_+(S_M)} \cap \overline{\pi_+(S_{M'})}$ equals $\overline{\pi_+(S_{M''})}$, where $S_{M''}$ lies in the closure of both S_M and $S_{M'}$.

  Arkani-Hamed-Trnka '13,..., Lukowski-Parisi-Williams '20
- **Goal:** To find good triangulations of $\mathcal{A}_{n,k,m}$.

Examples

- $k = 1$: a cyclic polytope with n vertices in $\mathbb{P}^m$. Arkani-Hamed, Trnka '13. Its triangulations are governed by the secondary polytope.





- $n = m + k$: positive Grassmannian $\mathrm{Gr}_+(k, n)$ is a simplex; it is decomposed into positroid cells. Postnikov '06

Examples

- The main case of interest is $m = 4$; The BCFW recurrence for computing scattering amplitudes is reformulated as giving a **triangulation** of $\mathcal{A}_{n,k,4}$.
- The cases $m = 1, 2$ are studied as toy models for $m = 4$.
  - $m = 1$: homeomorphic to the bounded complex of the cyclic hyperplane arrangement. Karp-Williams '19
  - $m = 2$: related to hypersimplex and tropical Grassmannain. Lukowski-Parisi-Williams '20
- Inspired by **parity conjugation** in quantum field theory.
- For $\ell = n - m - k$ there is a bijection between triangulations of $\mathcal{A}_{n,\ell,m}$ and triangulations of $\mathcal{A}_{n,k,m}$. Galashin-Lam '18

Theorem (M-Monin-Parisi '20)

There is a well-structured notion of secondary amplituhedra for $\ell = n - m - 1$.

Positive geometry

- Introduced by Arkani-Hamed-Trnka '14, Lam '14, Arkani-Bai-Lam '17
- A **positive geometry** is recursively defined as a pair $(X, X_{\geq 0})$ s.t. every irreducible component of $\overline{\partial X_{\geq 0}}$ is a positive geometry.
- A d-dim positive geometry is a pair $(X, X_{\geq 0})$, where
    - X is a d-dim normal, irreducible, **complex projective variety**
    - $X_{\geq 0} \subset X(\mathbb{R})$ is a **semialgebraic subset**
    - $X_{>0} = \text{int}(X_{\geq 0})$ is a d-dim **oriented real manifold**

  with a unique rational differential top-form $\Omega(X_{\geq 0})$ on X s.t.
    - its poles are all simple and are exactly along the boundary of $X_{\geq 0}$
    - its singularities are recursive, i.e. at every boundary component B

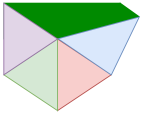
  $$\text{Res}_B \Omega(X_{\geq 0}) = \Omega(B_{\geq 0}).$$

- $\Omega(X_{\geq 0})$ is called the **canonical form**.
- **Application:** $\Omega(X_{\geq 0})$ is a physical observable.
- Polytopes, positive Grassmannian, amplituhedron, associahedron

Properties of positive geometry

- A finite set of positive geometries $(S, S_{\geq 0})$ **subdivides** $(X, X_{\geq 0})$ if
  - $\bigcup S_{\geq 0} = X_{\geq 0}$ and $S_{>}0 \subset X_{>0}$
  - The interiors $S_{>}0$ are pairwise disjoint
  - The orientations of $S_{>}0$ and $X_{>0}$ are compatible
- **Property 1.** Given a **subdivision** $\mathcal{C} = \{(S, S_{\geq 0})\}$ of $(X, X_{\geq 0})$ we have:

$$\Omega(X_{\geq 0}) = \sum \Omega(S_{\geq 0})$$



- **Property 2.** Given a diffeomorphism $\pi : (X, X_{\geq 0}) \to (Y, Y_{\geq 0})$, the canonical form $\Omega(Y_{\geq 0})$ is the pushforward of $\Omega(X_{\geq 0})$ by π.
- Given a subdivision $\mathcal{C} = \{S_M\}$ of $\mathcal{A}_{n,k,m}$ we have

$$\Omega(S_M) = \Omega(\pi(S_M)) \quad \text{and} \quad \Omega(\mathcal{A}_{n,k,m}) = \sum \Omega(S_M)$$

Different subdivisions of $\mathcal{A}_{n,k,m}$ lead to different ways to compute $\Omega(\mathcal{A}_{n,k,m})$.

Outlook: Geometry of subdivisions of $\mathcal{A}_{n,k,m}$

Question

Is there a notion of secondary amplituhedron encoding its subdivisions?

Motivated by **fiber polytopes**' construction of Billera-Sturmfels '92

- We extended the projection

$$\pi_+ : \mathrm{Gr}_+(k,n) \to \mathcal{A}_{n,k,m}$$

to a rational map

$$\pi : \mathrm{Gr}(k,n) \dashrightarrow \mathrm{Gr}(k,k+m).$$

and explicitly parametrize the fibers $(\pi^{-1}(Y), \pi_+^{-1}(Y))$ for $Y \in \mathcal{A}_{n,k,m}$.

- The **fiber volume form**: a top-form $\omega_{n,k,m}(Y)$ on the fiber.
- $\Omega(\mathcal{A}_{n,k,m})$ is the sum of certain residues of the forms $\omega_{n,k,m}(Y)$.
- When $\omega_{n,k,m}(Y)$ has only poles along some hyperplanes, we can use the classical Jeffrey-Kirwan residue to compute $\Omega(\mathcal{A}_{n,k,m})$.

Polytopes and their conjugates

- Let $k = 1$, $\ell = n - m - 1$ and $\mathcal{A} = \mathcal{A}_{n,1,m}$ or $\mathcal{A}_{n,\ell,m}$.
- The fiber volume forms $\omega_{n,k,m}(Y)$ have only poles along some hyperplane arrangement (the denominator is a product of linear forms).
- Let $\mathcal{F}_{\mathcal{A}}(Y)$ be the chamber fan on the normal rays of these hyperplanes.
- The structure of the fan $\mathcal{F}_{\mathcal{A}}(Y)$ is independent of Y and its cones correspond to positroidal subdivisions of the amplituhedron.

Theorem (M-Monin-Parisi '20)

There is a well-structured notion of secondary amplituhedra for $\ell = n - m - 1$.

Question

Is there a notion of secondary amplituhedron in general?

Reference

- **Triangulations and Canonical Forms of Amplituhedra: a fiber-based approach beyond polytopes**

  Mohammadi-Monin-Parisi (arXiv preprint arXiv:2010.07254)

Thank you for your attention!