Operads on noncrossing partitions

Loïc Foissy

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Let $X$ be a random variable; for any $n$, $m_n$ is the $n$-th moment of $X$ and $k_n$ is the $n$-th free cumulant of $X$. These two families are related:

$$m_n = \sum_{Q \in \text{NCP}(n)} \prod_{\pi \in Q} k_{|\pi|},$$

where $\text{NCP}(n)$ is the set of noncrossing partitions on $[n]$.

The aim is to "understand" this formula with a formalism of cointeracting bialgebras, using operads and related objects.
We denote by $\text{NCP}(n)$ the set of noncrossing partitions on $[n]$. They are represented by diagrams:

\begin{align*}
\text{NCP}(0) &= \{\emptyset\}, \\
\text{NCP}(1) &= \{\|\}, \\
\text{NCP}(2) &= \{\|, \_\}, \\
\text{NCP}(3) &= \{\|\|, \_\|, \|\_\}, \\
\text{NCP}(4) &= \left\{ \begin{array}{c}
\|\|\|, \_\|\|, \|\|\|, \|\_\|, \|\_\|, \_\|\|, \_\|\|, \|\|, \|\|, \_\|, \_\|, \_\|, \_\|, \_\| \end{array} \right\}.
\end{align*}
First operadic structure: insertion of noncrossing permutations in gaps of a noncrossing partition

Any element \( p \in \text{NCP}(n) \) has \( n + 1 \) gaps to insert other noncrossing partitions, from left to right: the first one on the left of the noncrossing partition, the last one on the right.

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\]
Proposition

With this composition, the sequence $\mathcal{NC}\mathcal{P}_0(n) = \mathbb{K}\mathcal{N}\mathcal{C}\mathcal{P}(n - 1)$ is a nonsymmetric operad denoted by $\mathcal{N}\mathcal{C}\mathcal{P}_0$.

$$\diamond : \bigoplus_{n=1}^{\infty} \mathcal{N}\mathcal{C}\mathcal{P}_0(n) \otimes \mathcal{N}\mathcal{C}\mathcal{P}_0^\otimes n \rightarrow \mathcal{N}\mathcal{C}\mathcal{P}_0.$$  

Unit of this operad: $\emptyset \in \mathcal{N}\mathcal{C}\mathcal{P}(0) \subseteq \mathbb{K}\mathcal{N}\mathcal{C}\mathcal{P}(1)$. 
Proposition

Algebras over $\mathcal{NC}\mathcal{P}_0$ are vector spaces $V$ given families of maps $\langle -, - \rangle : V^\otimes n \rightarrow V$ for any $n \geq 1$ such that:

$$\langle x_1, \ldots, x_{m-1}, \langle x_m, \ldots, x_{m+n-1} \rangle \rangle$$

$$= \langle \langle x_1, \ldots, x_m \rangle, x_{m+1}, \ldots, x_{m+n-1} \rangle.$$
Bialgebras from a nonsymmetric operad: If \((P, \circ)\) is a nonsymmetric operad, under finite-dimensional conditions, and if \(P(0) = (0)\),

\[
\circ : \bigoplus_{n=1}^{\infty} P(n) \otimes P^{\otimes n} \rightarrow P,
\]

\[
\Delta = \circ^* : P^* \rightarrow \bigoplus_{n=1}^{\infty} P^*(n) \otimes P^{* \otimes n} \subseteq P^* \otimes T(P^*).
\]

\(\Delta\) can be extended as an algebra morphism from \(T(P^*)\) to \(T(P^*) \otimes T(P^*)\). Then:

**Proposition**

\(T(P^*), m_{\text{conc}}, \Delta\) is a graded bialgebra. The elements of \(P^*(n)\) are homogeneous of degree \(n - 1\).
We identify $\mathcal{NCP}_0^*$ and its dual, the basis $\text{NCP}_0$ being now self-dual. The obtained coproduct is the following:

$$
\Delta_0(\emptyset) = \emptyset \otimes \emptyset,
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$$

$T(\mathcal{NCP}_0)$ is a bialgebra. It is graded by the number of blocks. Its counit is given by

$$
\forall P \in \text{NCP}_0, \quad \varepsilon(P) = \delta_{P,\emptyset}.
$$
Let us identify $\emptyset$ and the unit of $T(\mathcal{NCP}_0)$:

\[
\Delta(\mathbb{1}) = \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1},
\]

\[
\Delta(\cup') = \cup' \otimes \mathbb{1} + \mathbb{1} \otimes \cup',
\]

\[
\Delta(\downarrow) = \downarrow \otimes \mathbb{1} + \cup \otimes \mathbb{1} + \mathbb{1} \otimes \downarrow,
\]

\[
\Delta(\downarrow') = \downarrow' \otimes \mathbb{1} + \cup \otimes \mathbb{1} + \mathbb{1} \otimes \downarrow'.
\]

$T(\mathcal{NCP})$ is a graded, connected Hopf algebra. Its counit is given by

\[
\forall P \in \mathcal{NCP}, \quad \epsilon(P) = 0.
\]
$T(\mathcal{NCP})$ is a graded, connected Hopf algebra. Its coproduct is given by admissible cuts, that is to say ways to separate the blocks into a lower and an upper part:

$$\Delta(P) = \sum_{(L,U) \in \text{cut}(P)} L \otimes \dot{U}.$$ 

**Corollary**

The morphism from $T(\mathcal{NCP})$ sending a noncrossing partition to its plane forest is a surjective Hopf algebra morphism from $(T(\mathcal{NCP}, m_{\text{conc}}, \Delta)$ to the noncommutative Connes-Kreimer Hopf algebra of plane rooted trees.
More structure for almost free:

\[ \Delta_<(P) = \sum_{(L,U) \in \text{cut}(P), 1 \in L} L \otimes \dot{U}, \]

\[ \Delta_>(P) = \sum_{(L,U) \in \text{cut}(P), 1 \in U} L \otimes \dot{U}. \]

Then:

\[ (\Delta \otimes \text{Id}) \circ \Delta_> = (\text{Id} \otimes \Delta_>) \circ \Delta_>, \]

\[ (\Delta_> \otimes \text{Id}) \circ \Delta_<= (\text{Id} \otimes \Delta_<) \otimes \Delta_>, \]

\[ (\Delta_< \otimes \text{Id}) \circ \Delta_< = (\text{Id} \otimes \Delta) \circ \Delta_<. \]

This is a unshuffle bialgebra (or codendriform bialgebra).
As a consequence, the dual algebra $T(\mathcal{NCP})^*$ is a dendriform algebra, with convolution product $* = < + >$.

**An example**

Let $\kappa$ be the infinitesimal character on $T(\mathcal{NCP})$ such that for any noncrossing partition $P$,

$$\kappa(P) = \begin{cases} k_n & \text{if } P \text{ has only one block of size } n, \\ 0 & \text{otherwise.} \end{cases}$$

There exists a unique $\phi \in T(\mathcal{NCP})^*$ such that

$$\phi = \varepsilon + \kappa < \phi.$$ 

Then $\phi$ is a character and for any noncrossing partition $Q$:

$$\phi(Q) = k_Q.$$
Second operadic structure: block substitution

We want to replace any block of a noncrossing partition by another noncrossing partition:

\[
\begin{array}{c}
\text{old partition} \\
\begin{array}{c}
\text{new partition}
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\]

\[
\begin{array}{c}
\text{old partition} \\
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\]

First technical difficulty: how to know which block is replaced by each noncrossing partition?
We want to replace any block of a noncrossing composition by another noncrossing compositions:

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The numbers indicate the index of the blocks in the noncrossing partitions.
We want to replace any block of a noncrossing composition by another noncrossing compositions:

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Second technical difficulty: some compositions are not possible. Each block has a color (its size); we can only substitute to a block of color \( n \) a noncrossing composition of degree \( n \).
Let us denote by $\text{NCC}(n; k_1, \ldots, k_p)$ the set of noncrossing compositions on $[n]$ with $p$ blocks, the $i$-th block of size $k_i$. The composition is defined from

$$\text{NCC}(n; k_1, \ldots, k_p) \otimes \text{NCC}(k_1; \ell_1) \otimes \ldots \otimes \text{NCC}(k_p; \ell_p)$$

to

$$\text{NCC}(k_1 + \ldots + k_p; \ell_1, \ldots, \ell_p).$$
Proposition

With this composition, the sequence
$$\mathcal{NCC}(n; k_1, \ldots, k_p) = \mathcal{KNCC}(n; k_1, \ldots, k_p)$$
is a colored operad.

Partial unit $\in \mathcal{NCC}(n; n)$: $l_n = ([n])$.

$$l_1 = 1, \quad l_2 = \emptyset, \quad l_3 = \mathcal{C}, \quad l_4 = \mathcal{C}\mathcal{C} \ldots$$
\( \mathcal{NCC} \) can also be seen as a non unitary operad, considering:

\[
\mathcal{NCC}(p) = \bigoplus_{n,k_1,\ldots,k_p \geq 1} \mathcal{NCC}(n; k_1, \ldots, k_p).
\]

Third technical difficulty: this is not finite-dimensional.

Fourth technical difficulty: if we want a unit, we have to consider infinite sums as

\[
I = \sum_{n \geq 1} I_n.
\]
Let us consider the bialgebra attached to this colored operad. We shall identify $\mathcal{NCC}^\ast(k_1, \ldots, k_p)$ with its dual, the basis $\mathcal{NCC}(k_1, \ldots, k_p)$ being self-dual.

1. $T(\mathcal{NCC})$ is a bialgebra.
2. Its abelianized algebra $S(\mathcal{NCC})$ is also a bialgebra.
3. The subalgebra $S(\text{inv}_\mathcal{G}(\mathcal{NCC}))$ is a subbialgebra of $S(\mathcal{NCC})$, where $\text{inv}_\mathcal{G}$ is the subspace of invariants of $\mathcal{NCC}$ under the action of symmetric groups by permutations of the blocks. This is identified with $S(\mathcal{NCP})$.

The coproducts of these bialgebras are all denoted by $\delta$. 
Examples of coproducts in $S(\mathcal{NCP})$:

\[
\begin{align*}
\delta(11) &= 11 \otimes 1 \cdot 1 + \underline{\cdot} \otimes 11 \\
\delta(111) &= 111 \otimes 1 \cdot 1 + 1\underline{\cdot} \otimes 11 \\
\delta(11) &= 1\underline{\cdot} \otimes 1 \cdot 1 + 11 \otimes 1\underline{\cdot} + (1\underline{\cdot} + 1\underline{\cdot}) \otimes 1\underline{\cdot} + \underline{\cdot} \otimes 11 \\
\delta(111) &= 111 \otimes 1 \cdot 1 + 1\underline{\cdot} \otimes 11 + 1\underline{\cdot} \otimes 1\underline{\cdot} + \underline{\cdot} \otimes 1\underline{\cdot}
\end{align*}
\]
The finest partitions have the longest formulas for the coproduct:

\[
\begin{align*}
\delta(\, & ) = \, | | \otimes \, | | + \, | | \otimes \, | \cdot |
\\
\delta(\, &&) = \, | | | \otimes \, | | | + (\, | | \otimes | | + | | \otimes | |) \otimes \, | \cdot | \cdot | + | | | \otimes \, | \cdot | \cdot | \cdot |
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\end{align*}
\]
Partial order on noncrossing partitions

Let $P$, $Q$ be two noncrossing partitions. Then $P \preceq Q$ if $P$ is a refinement of $Q$. 
Working similarly with noncrossing compositions,

\[ P \leq Q \iff \exists P_1, \ldots, P_n, P = Q \circ (P_1, \ldots, P_n). \]

\((P_1, \ldots, P_n)\) is unique up to their order, and we put in \(S(\mathcal{NPCP})\):

\[ P/Q = P_1 \ldots P_n. \]

Then, in \(S(\mathcal{NPCP})\):

\[ \delta(P) = \sum_{Q \succeq P} Q \otimes P/Q. \]

We recover an incidence algebra in the sense of Schmitt.
The coproduct $\delta$ induces a second convolution $\ast$ product on $S(\mathcal{NCP})^\ast$.

$$m_n = \sum_{Q \succeq J_n} k_{J_n/Q} = \zeta \ast k(J_n),$$

where $\zeta(P) = 1$ for any noncrossing partition and

$$J_n = \{\{1\}, \ldots, \{n\}\} = \underbrace{\ldots}_{n}.$$

$$J_1 = |, \quad J_2 = ||, \quad J_3 = |||, \quad J_4 = ||||, \ldots$$
Theorem

\((T(\mathcal{NC}P), m_{\text{conc}}, \Delta_\prec, \Delta_\succ)\) is a bialgebra in the category of right \((S(\mathcal{NC}P), m, \delta)\)-comodules, with the coaction

\[ \rho = (\text{Id} \otimes \pi) \circ \delta : T(\mathcal{NC}P) \longrightarrow T(\mathcal{NC}P) \otimes T(\mathcal{NCC}) \]
\[ \longrightarrow T(\mathcal{NC}P) \otimes S(\mathcal{NC}P). \]

\[ (\Delta_\prec \otimes \text{Id}) \circ \rho = m_{1,3,24} \circ (\rho \otimes \rho) \circ \Delta_\prec, \]
\[ (\Delta_\succ \otimes \text{Id}) \circ \rho = m_{1,3,24} \circ (\rho \otimes \rho) \circ \Delta_\succ, \]
\[ (\Delta \otimes \text{Id}) \circ \rho = m_{1,3,24} \circ (\rho \otimes \rho) \circ \Delta. \]
For any infinitesimal character $\kappa$, let us denote by $\mathcal{E}_<(\kappa)$ the unique element $\phi$ of $T(\mathcal{NCP})^*$ such that

$$\phi = \varepsilon + \kappa < \phi.$$ 

From the dendriform axioms, it is a character of $T(\mathcal{NCP})$ or of $S(\mathcal{NCP})$.

The map $\mathcal{E}_<$ is the left half-exponential.
An application

Let $e$ be the infinitesimal character on $S(NCP)$ defined by:

$$e(P) = \begin{cases} 
1 & \text{if } P \text{ has one block}, \\
0 & \text{otherwise}.
\end{cases}$$

Then $\pi_1$ is the block of $P$ containing 1,

$$\mathcal{E}_<(e)(P) = 1 \times \mathcal{E}_<(e)(P\backslash\{\pi_1\}).$$

Hence, for any noncrossing partition $P$,

$$\mathcal{E}_<(e)(P) = 1 = \zeta(P).$$
An application

The coaction \( \rho \) induces an action \( \preceq \) of the monoid of character \( M \) of \( (S(NCP), m, \delta) \) on \( (T(NCP)^*, \star) \).

**Proposition**

Let \( \kappa \) be an infinitesimal character and \( \psi \) be a character of \( S(NCP) \). We denote by \( K \) the character such that for any noncrossing partition \( P \), \( K(P) = \kappa(P) \). Then:

\[
\phi = \mathcal{E}_\kappa(\kappa) \iff \phi = \zeta \preceq K.
\]

\( K \) is the unique character such that \( e \preceq K = \kappa \).
Let $\psi = \zeta \leftarrow K$. Then:

\[
\begin{align*}
\psi &= \mathcal{E}_<(e) \leftarrow K \\
&= (\epsilon + e < \zeta) \leftarrow K \\
&= \epsilon \leftarrow K + (e < \zeta) \leftarrow K \\
&= \epsilon + (e \leftarrow K) < (\zeta \leftarrow K) \\
&= \epsilon + \kappa < \psi.
\end{align*}
\]

So $\psi = \mathcal{E}_<(\kappa)$. 
Consequently, if $\psi = \mathcal{E}_<(\kappa)$:

$$\psi(J_n) = \sum_{Q \in \text{NCC}(n)} \prod_{\pi \in Q} k(|J|_\pi).$$
Kurusch Ebrahimi-Fard, Loïc Foissy, Joachim Kock, Frédéric Patras:

*Operads of (noncrossing) partitions, interacting bialgebras, and moment-cumulant relations*

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Thank you for your attention!