$p(x)$-Calderón's problem in 1D

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$p(x)$-Calderón's problem

$$\operatorname{div}\left(\gamma(x)\,|\nabla u|^{p(x)-2}\,\nabla u\right) = 0$$

Exponent $p(\cdot)$ is known and we wish to recover the unknown γ from DN map,

$$\Lambda_\gamma(m) = \int_I \gamma^{-1/(p(x)-1)} K_m^{p(x)/(p(x)-1)} \mathrm{d}x,$$

where $K_m \in \mathbb{R}_+$ depends on γ:

$$m = \int_I \gamma^{-1/(p(x)-1)} K_m^{1/(p(x)-1)} \mathrm{d}x.$$

Main theorem

- Bounded open interval I. $L^2(I, \sigma(p))$ is the L^2 space on I with the coarsest σ-algebra $\sigma(p)$ that makes p measurable.
- $P\colon L^2(I) \to L^2(I, \sigma(p))$ is the orthogonal projection.

Theorem 1. *The nonlinear projection*

$$\widetilde{P}(\gamma)(x) = \left(P\left(\gamma^{-1/(p-1)}\right)(x)\right)^{-(p(x)-1)}$$

can be constructively recoved from the DN map.

Examples

- If p is **injective**, then P is the identity and γ **can be fully recovered**.
- If γ is $\sigma(p)$-measurable, then $\widetilde{P}(\gamma) = \gamma$.
- The projection of f to $L^2(I, \sigma(p))$ is the conditional expectation $\mathbb{E}(f|\sigma(p))$.
- If p is constant on a set, than only an average of γ to a power can be recovered there.

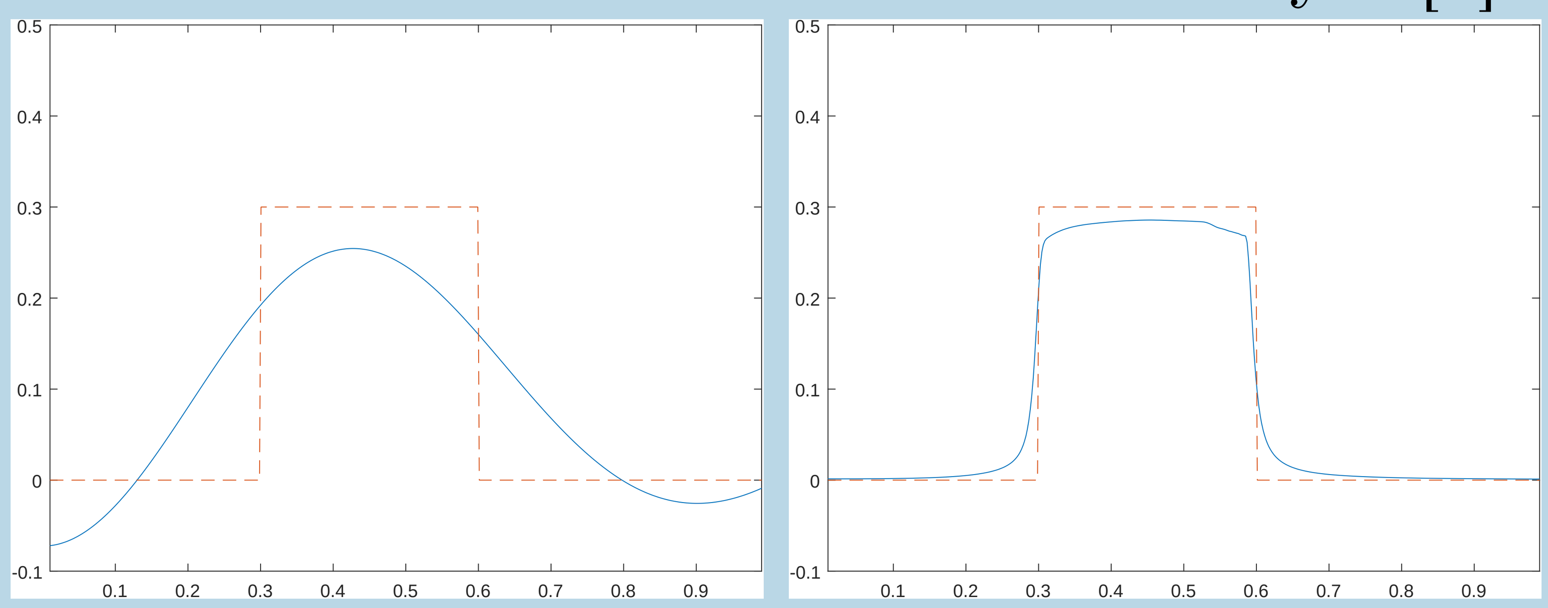
References

[1] T. Brander, D. Winterrose: *Variable exponent Calderón's problem in one dimension*, to appear in Annales Academiæ Scientiarum Fennicæ. `https://arxiv.org/abs/1808.04168`

[2] T. Brander, J. Ilmavirta, T. Tyni: *Optimal recovery of a radiating source with multiple frequencies along one line*, preprint `https://arxiv.org/abs/1905.08028`.

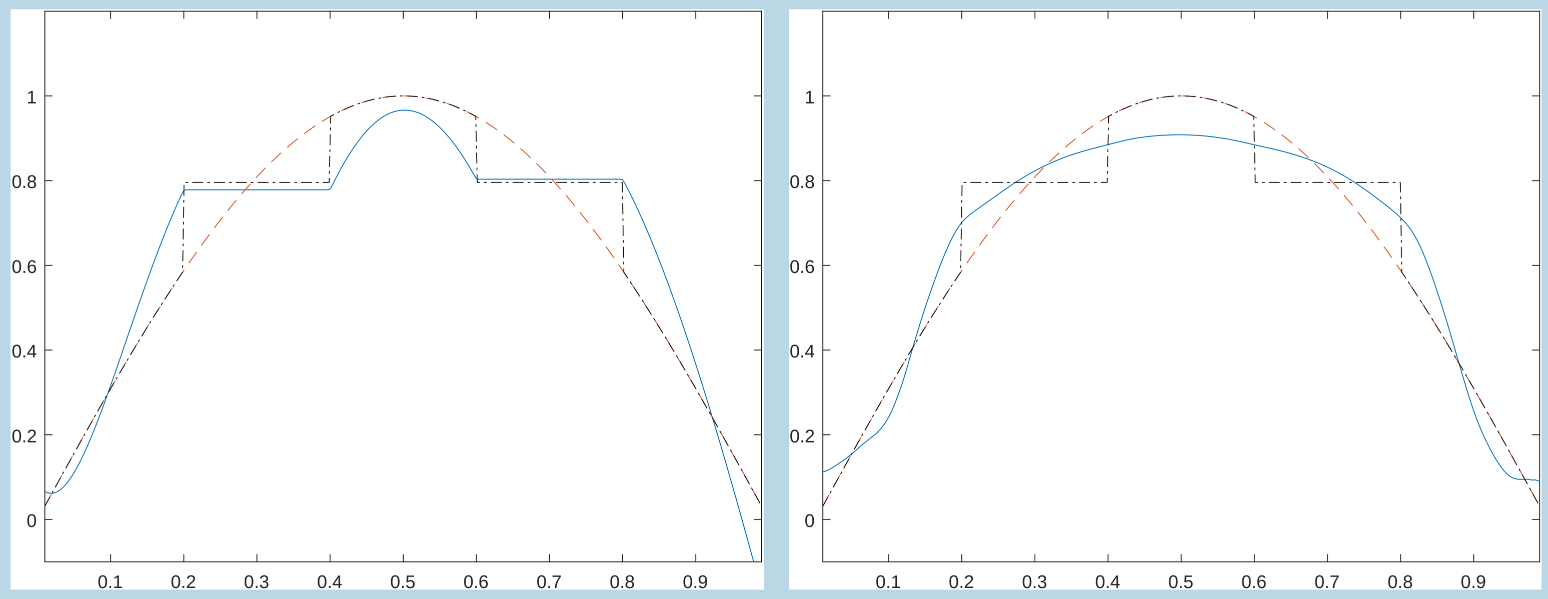
Reconstruction, injective p

The reconstructions are for a slightly simpler problem with Joonas Ilmavirta and Teemu Tyni [2].



Unknown γ_0 (dashed red line) and the numerical solution γ (solid blue line) with 0.5% noise level. Tikhonov-solution (left), TV-solution (right).

Reconstruction, non-injective p



Unknown γ_0 (dashed red line), γ_0 averaged over regions where p is constant (black dot-dash line) and the numerical solution γ (solid blue line) with 0.5% noise level. Tikhonov-solution (left), TV-solution (right).

Steps in the proof

1. $$\int_I \gamma^{-1/(p(x)-1)} \mathrm{d}x$$
   can be recovered as the unique fixed point of the DN map. This is the constant p result.
2. The derivatives of the DN map give for all n
   $$\int_I \gamma^{-1/(p(x)-1)} \left(\frac{1}{p(x)-1}\right)^n \mathrm{d}x.$$
3. The closure of span $\left\{\left(\frac{1}{p(x)-1}\right)^n ; n \in \mathbb{N}\right\}$ in L^2 is $L^2(I, \sigma(p))$.
4. $\gamma^{-1/(p(x)-1)}$ is coupled with $L^2(I, \sigma(p))$. This gives information in the dual, $L^2(I, \sigma(p))$.