Convergence analysis of *spatially dependent* Ostrovsky Hunter Equation

Neelabja Chatterjee

(Joint work with Prof. Nils Henrik Risebro)

Universitetet i Oslo, Department of Mathematics

The Equation

For some finite $T > 0$,

$$\begin{aligned} u_t + f(x,u)_x &= P[u](x,t) \text{ for } x \in [0,1],\ t \in (0,T], \\ u(x,0) &= u_0(x) \text{ for } x \in [0,1], \\ u(0,t) &= u(1,t) \text{ for } t \in (0,T] \ (\rightsquigarrow \text{periodic B.C. in space}), \end{aligned} \tag{1}$$

where the non-local source term is given by

$$P[u](x,t) := \int_0^x u(y,t)\,dy - \int_0^1 \int_0^y u(z,t)\,dz\,dy. \tag{2}$$

Associated Assumptions

We will have few assumptions on the ***flux*** and ***initial data*** in (1)

- **(A1)** $f(x,u) \in C^2_{\text{loc}}([0,1] \times \mathbb{R})$. In particular, (for e.g. $f(x,u) = e^{\sin(2\pi x)}\frac{u^2}{2}$)
  - **(A1.a)** $f(x,u)$ is Lipschitz continuous in x and *locally* Lipschitz continuous in u,
  - **(A1.b)** $f_x(x,u)$ is Lipschitz continuous in x and *locally* Lipschitz continuous in u.
- **(B1)** $u_0(x) \in \text{BV}([0,1])$ and $\int_0^1 u_0(x)\,dx = 0$ ($\rightsquigarrow$ zero mean condition),

Main Objectives

1. To derive and analyze a numerical method for the equations (1) satisfying the above assumptions.
2. To show the convergence of the numerical scheme to an entropy solution of the equation (1), hence concluding *existence*.
3. To show some type of *stability* result of the scheme, concluding *uniqueness* of the entropy solution.
4. To provide some explicit convergence rate.

Main Result

- **Theorem [3]:** With the assumptions stated above, **there exists a unique entropy solution**, for the equation (1) given by our scheme (described below) and the scheme converges to this solution with an order $\frac{1}{2}$.

Notion of Solution

- ***Entropy Solution:*** *(see [1])* $u \in C([0,T]; L^1((0,1))) \cap L^\infty([0,1] \times [0,T])$ is an *entropy solution* of the equation (1) if for all $k \in \mathbb{R}$

$$\begin{aligned} L_{\text{Kr}}(u,k,\phi) := \int_0^T \int_0^1 \Big[\eta(u,k)\phi_t + q(x,u,k)\phi_x + \operatorname{sign}(u-k) f_x(x,k)\phi + \operatorname{sign}(u-k)P[u]\phi\Big]\,dx\,dt \\ + \int_0^1 \eta(u_0(x),k)\phi(x,0)\,dx - \int_0^1 \eta(u(x,T),k)\phi(x,T)\,dx = 0, \end{aligned}$$

for all non-negative, spatially 1-periodic test fucntions $\phi \in C^1_c([0,1] \times [0,T])$, where $\eta(u,k) := |u-k|$ and $q(x,u,k) := \operatorname{sign}(u-k)(f(x,u) - f(x,k))$.

Discretization and Scheme

- **Spatial mesh parameter:** $\Delta x := \frac{1}{N}$ and **Temporal mesh parameter:** $\Delta t := \frac{T}{M+1}$, for $N, M \in \mathbb{N}$.
- **Mesh:** (Finite volume method) $\rightarrow$ ***(Spatial:)*** for $j = 0,1,\cdots,N$, $x_{j+\frac{1}{2}} := j\Delta x$ and for $j = 1,\cdots,N$, $x_j := x_{j+\frac{1}{2}} - \frac{\Delta x}{2}$; ***(Temporal:)*** for $n = 0,1,\cdots,M+1$, $t^n := n\Delta t$.
- **Notation:** $I_j := [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$, $I^n := [t^n, t^{n+1})$, $I^n_j := I_j \times I^n$, and $\lambda = \frac{\Delta t}{\Delta x}$.
- **The scheme:**

$$u^{n+1}_j = u^n_j - \lambda\Big(F^n_{j+\frac{1}{2}} - F^n_{j-\frac{1}{2}}\Big) + \Delta t P^n_j := G_j(u^n_{j-1}, u^n_j, u^n_{j+1}) + \Delta t P^n_j, \tag{3}$$

where

$$u^0_j := u(x_{j+\frac{1}{2}}, 0) \approx \frac{1}{\Delta x}\int_{I_j} u_0(x)\,dx, \tag{4}$$

$$P^n_j := \Delta x\Big(\sum_{i=1}^{j-1} u^n_i + \frac{1}{2}u^n_j\Big) - (\Delta x)^2 \sum_{l=1}^{N}\Big(\sum_{i=1}^{l-1} u^n_i + \frac{1}{2}u^n_l\Big),\ \Big(\implies \sum_{j=1}^{N} P^n_j = 0\Big) \tag{5}$$

and

$$F^n_{j+\frac{1}{2}} = F(x_{j+\frac{1}{2}}, u^n_j \uparrow, u^n_{j+1} \downarrow), \text{ with } u^n_{N+1} = u^n_1 \text{ and } u^n_N = u^n_0. \tag{6}$$

- **Remark:** Conditions (5) + (6) + *spatial periodicity* + 'zero mean property' $\implies$ 'MASS' conservation!!
- **For the sake of simplicity:** assume $f_u(x,u) \geq 0$. General case : Similar but lengthy calculations!!

Convergence and Convergence Rate (Proof of Main Result)

A priori estimates [NC-Risebro [3], '18]

- **L^∞ bound :**

$$\|u^n\|_\infty \leq e^{2T}\|u^0\|_\infty.$$

- **T.V. bound :**

$$|u^n|_{\text{BV}([0,1])} \leq e^{C_f T}|u^0|_{\text{BV}([0,1])} + C_f\Big(e^{C_f T} - 1\Big),$$

where the constant C_f depends on the *Lipschitz* constants of f_x in x and u.

- **Time continuity :**

$$\Delta x \sum_{j=1}^{N}\left|\frac{u^{n+1}_j - u^n_j}{\Delta t}\right| \leq C_f\Big(|u^0|_{\text{BV}([0,1])} + 1\Big).$$

Compactness, hence *Existence*

- Defining $u_{\Delta x}(x,t) := \Sigma_{j,n}\, u^n_j \chi_{I^n_j}(x,t)$, application of *a priori* estimates of previous subsection + Helly's theorem + Arzela Ascoli's theorem yields:
- **Theorem [NC-Risebro [3], '18]:** Let $\{u_{\Delta x}\}_{\Delta x > 0}$ be the family of approximate solutions obtained from the scheme (3) with λ chosen such that G_j is monotone for all j and $t^n \leq T$. Then there exists a subsequence $\{\Delta x_k\}_{k \in \mathbb{N}}$ with $\Delta x_k \to 0$ as $k \to \infty$ and a function $u \in C([0,T]; L^1([0,1]))$ such that $u_{\Delta x_k} \to u$ in $C([0,T]; L^1([0,1]))$.
- Denoting $D^t_+ a^n_j := \frac{a^{n+1}_j - a^n_j}{\Delta t}$ and $a_j - a_{j-1} =: \Delta x D_- a_j$,
- ***Discrete* Entropy Inequality [NC-Risebro [3] '18]:**

$$D^t_+ \eta(u^n_j, k) + D_- Q_{j+\frac{1}{2}}(u^n_j, k) + \operatorname{sign}(u^{n+1}_j - k) D_- F_{j+\frac{1}{2}}(k) - \operatorname{sign}(u^{n+1}_j - k) P^n_j \leq 0,$$

where $Q_{j+\frac{1}{2}}(u,k) := F_{j+\frac{1}{2}}(u \vee k) - F_{j+\frac{1}{2}}(u \wedge k) = \operatorname{sign}(u-k)\Big(F_{j+\frac{1}{2}}(u) - F_{j+\frac{1}{2}}(k)\Big)$.

- ***Existence* of Entropy Solution: [NC-Risebro [3] '18]**
  Theorem: If the initial data $u_0 \in \text{BV}([0,1])$ satisfies the zero-mean property and λ satisfies the CFL condition $\lambda F_u \leq 1$, then $u =: lim_{k\to\infty} u_{\Delta x_k}$ is **an** entropy solution of the equation (1).

Stability and Convergence Rate

- Denoting

$$\Lambda_{\epsilon,\epsilon_0}(u,v) := \int_0^T \int_0^1 L(u, v(y,s), \phi_{\epsilon,\epsilon_0(\cdot,\cdot,y,s)})\,dy\,ds,$$

- **Kuznetsov type Lemma [NC-Risebro [3], '18]:** (see [2]) If u be an entropy solution of the equation (1) with the associated initial data $u_0 \in \text{BV}([0,1])$, then for any $v \in L^\infty([0,T]; L^1([0,1])) \cap L^\infty(\Pi_T)$, then we have

$$\begin{aligned} \|u(\cdot,T) - v(\cdot,T)\|_1 \leq e^{2T}\|u_0(\cdot) - v_0(\cdot)\|_1 + (e^{2T} - 1) \times \Big[-\Lambda_{\epsilon,\epsilon_0}(v,u) + \frac{1}{2}\Big(\mu(v(\cdot,T),\epsilon) + \mu(u(\cdot,T),\epsilon) \\ + \mu(v_0(\cdot),\epsilon) + \mu(u_0(\cdot),\epsilon)\Big) + C_{f,T}(\epsilon + T \sup_{t \in [0,T]} \mu(v(\cdot,t),\epsilon) + \nu(v,\epsilon_0))\Big]. \end{aligned}$$

- Letting $\epsilon, \epsilon_0 \to 0$, in the lemma, we conclude for two entropy solutions u, v of equation (1) with correspondin initial data u_0, v_0, we have

$$\|u(\cdot,T) - v(\cdot,T)\|_1 \leq e^{2T}\|u_0(\cdot) - v_0(\cdot)\|_1 \ \rightsquigarrow \ \textit{Uniqueness}. \tag{7}$$

- Also, in the estimate $-\Lambda_{\epsilon,\epsilon_0}(u_{\Delta x}, u) \leq C_T\Big(\Delta t + \Delta x + \frac{\Delta t}{\epsilon_0} + \frac{\Delta x}{\epsilon}\Big)$, choosing $\epsilon = \epsilon_0 = \sqrt{\Delta t} = C\sqrt{\Delta x}$, we get $\|u(\cdot,T) - u_{\Delta x}(\cdot,T)\|_1 \leq C_T\sqrt{\Delta x}$, ($\implies$ the scheme converges to **the entropy solution** with an order $\frac{1}{2}$).

Numerical example

To validate our theoretical result, we have performed numerical simulation with following two initial data, reference mesh being $N = 8192$,

$$u_0(x) = \begin{cases} \frac{1}{6}(x - \frac{1}{2})^2 + \frac{1}{6}(x - \frac{1}{2}) + \frac{1}{36} & , \quad \text{if} \quad 0 \leq x < \frac{1}{2} \\ \frac{1}{6}(x - \frac{1}{2})^2 - \frac{1}{6}(x - \frac{1}{2}) + \frac{1}{36} & , \quad \text{if} \quad \frac{1}{2} \leq x < 1, \end{cases} \tag{8}$$

and

$$u_0(x) = -0.05\cos(2\pi x), \text{ for } x \in [0,1], \text{ (see [4])}. \tag{9}$$

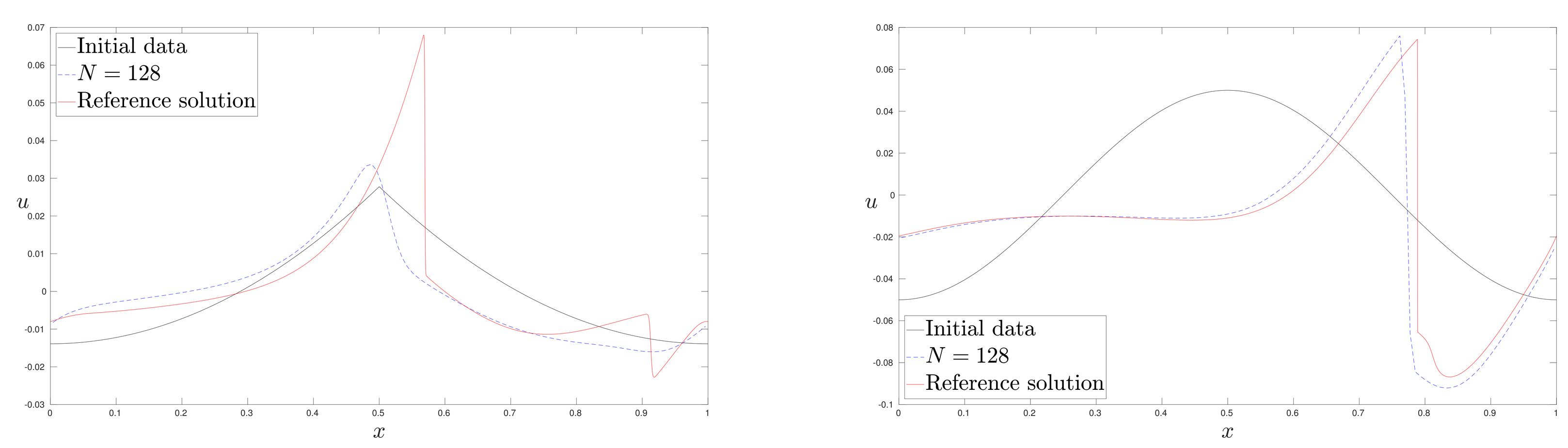

Figure 1: solutions on different meshes at $t = 36$ for initial data (8) (left) and for initial data (9) (right)

	Initial data (8)		Initial data (8)	
N	Error(%)	EOC	Error(%)	EOC
32	56.4		65.3	
64	40.9	0.5	39.5	0.7
128	26.4	0.6	21.8	0.9
256	15.5	0.8	11.4	0.9
512	7.8	1.0	5.9	1.0
1024	3.7	1.1	2.7	1.1
2048	1.6	1.2	1.2	1.2

Table 1: Relative errors and EOC in the L^1 distances at t = 36 for different initial data.

Future Projects

- Analysis of OH equation with *space-time dependent* flux,
- Analysis of OH equation using *operator splitting* method.

References

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