Error Bounds of Monotone Schemes for Strongly and Weakly Degenerate Nonlocal HJB Equation

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Introduction

We consider the nonlocal Hamilton-Jacobi-Bellman equation of the following form:

$$\sup_{\alpha\in\mathcal{A}}\Big\{f^\alpha(x)+c^\alpha(x)u(x)-\mathcal{I}^\alpha[u](x)\Big\}=0 \quad \text{in } \mathbb{R}^N, \quad (1)$$

where $\mathcal{A}$, set of all admissible controls, is a compact metric space. The integral operator $\mathcal{I}^\alpha$ is defined as

$$\mathcal{I}^\alpha[\phi](x):=\int_{\mathbb{R}^N\setminus\{0\}}\Big(\phi(x+\eta^\alpha(z))-\phi(t,x) -\eta^\alpha(z)\cdot\nabla_x\phi(x)\Big)\nu_\alpha(dz).$$

For each $\alpha\in\mathcal{A}$, ν_α (singular Lévy measure) is a non-negative Radon measure on $\mathbb{R}^N\setminus 0$ satisfying

$$\int_{|z|<1}|z|^2\nu_\alpha(dz)+\int_{|z|>1}\nu_\alpha(dz)<\infty.$$

Assumptions

We consider two different sets of assumptions for strongly and weakly degenerate case.

On strongly degenerate equations:

(**A**.1) For $\alpha\in\mathcal{A}$ the Lévy measures ν_α are symmetric. Furthermore, there is some $\sigma\in(0,2)$, a constant $C>0$, and density $\tilde\nu_\alpha(z)$ of $\nu_\alpha(dz)$ for $|z|<1$ satisfying

$$0\le\tilde\nu_\alpha(z)\le\frac{C}{|z|^{N+\sigma}}\qquad\text{for}\qquad |z|<1.$$

(**A**.2) $c^\alpha(x)\ge\lambda>0$, and $c^\alpha(x)$, $f^\alpha(x)$, and $\eta^\alpha(z)$ are continuous in α, x and z. In addition, there exists a constant $K>0$ such that for every α,

$$\|f^\alpha\|_1+\|c^\alpha\|_1+\|\eta^\alpha\|_0\le K.$$

(**A**.3) For each $\alpha\in\mathcal{A}$, the jump term η^α satisfies

$$\eta^\alpha(-z)=-\eta^\alpha(z)\quad\text{and}\quad|\eta^\alpha(z)|\le K_1|z|\quad\text{for}\quad|z|<1,$$

for some constant $K_1>0$.

On weakly degenerate equations:

In addition to the above assumptions on the data, we consider the following sets of assumptions for $\sigma>1$ in this case:

(**B**.1) **Weak-degeneracy:** There exists $\alpha_0\in\mathcal{A}$ and $c_1^{\alpha_0}>\delta$ for some $\delta>0$ such that the density from assumptions (**A**.1) satisfies

$$\tilde\nu_{\alpha_0}(z)\ge\frac{c_1^{\alpha_0}}{|z|^{N+\sigma}}\quad\text{for}\quad|z|<1.$$

(**B**.2) There exists $\beta>\sigma-1$ and a constant $K>0$ such that for every α we have $f^\alpha\in C^{1,\beta}(\mathbb{R}^N)$ and $\|f^\alpha\|_{1,\beta}\le K$.

(**B**.3) There exists a constant $K_2>0$ such that for any $|z|<1$ and for each $\alpha\in\mathcal{A}$ we have

$$|\eta^\alpha(z)-\eta^\alpha(0)-z|\le K|z|^2.$$

As the equation (1) *is fully nonlinear, the solution for this type of equations are interpreted through* ***'viscosity solution'*** *sense.*

Theorem(Regularity of solutions)

(*Strongly Degenerate*) Assume (**A**.1)-(**A**.3) for the equation (1) and denote $\lambda_0=\sup_{\alpha\in\mathcal{A}}(\|u\|_0\|c^\alpha\|_1+\|f^\alpha\|_1)$. If $\lambda>\lambda_0$ then the viscosity solution u of (1) is Lipschitz continuous.

(*Weakly Degenerate*)Assume (**A**.1)-(**A**.3) and (**B**.2)-(**B**.3) hold and let u be the unique viscosity solution of (1), then $(-\Delta)^{\frac{\sigma}{2}}[u]\in L^\infty(\mathbb{R}^N)$.

Remark 1. *The improved regularity structure in weakly degenerate case is still not sufficient to define the equation classically.*

Improved Monotone Difference Quadrature Schemes

we make the approximation in two steps.

Approximations of singular part near origin:

For sufficiently smooth function ϕ and $\delta>0$:

$$\begin{aligned}\mathcal{I}^\alpha[\phi](x)=&\int_{|z|<\delta}\Big(\phi(x+\eta^\alpha(z))-\phi(t,x)-\eta^\alpha(z)\cdot\nabla\phi(x)\Big)\nu_\alpha(dz)\\&+\int_{|z|>\delta}\Big(\phi(x+\eta^\alpha(z))-\phi(t,x)\Big)\nu_\alpha(dz)\\:=&\ \mathcal{I}^\alpha_\delta[\phi](x)+\mathcal{I}^{\alpha,\delta}[\phi](x).\end{aligned}$$

Under the assumptions (**A**.1)-(**A**.3), we get by Taylor's expansion

$$\mathcal{I}^\alpha_\delta[\phi](x)=tr[a^\alpha_\delta D^2\phi]+\mathcal{O}\big(\delta^{4-\sigma}\big),$$

where, $a^\alpha_\delta=\frac12\int_{|z|<\delta}\eta^\alpha(z)\eta^\alpha(z)^T\nu_\alpha(dz)$. The expression of a^α_δ guaranties that it would be a $N\times N$ constant *positive semi-definite matrix.* We approximate the equation (1) by replacing $\mathcal{I}^\alpha_\delta[\phi]$ with $tr[a^\alpha_\delta D^2\phi]$ and write

$$\sup_{\alpha\in\mathcal{A}}\Big\{f^\alpha(x)+c^\alpha(x)u(x)-tr[a^\alpha_\delta D^2u](x)-\mathcal{I}^{\alpha,\delta}[u](x)\Big\}=0.\quad(2)$$

Monotone discretization of approximate equation (2):

Define $i_h(\phi)(x)=\sum_{\mathbf{j}\in\mathbb{Z}^N}\phi(x_j)\omega_{\mathbf{j}}(x)$ where i_h *is linear or multilinear interpolant* and the weight function $\omega_{\mathbf{j}}\ge0$ with $\sum_{\mathbf{j}\in\mathbb{Z}^N}\omega_{\mathbf{j}}=1$. We approximate the term $tr[a^\alpha_\delta D^2\phi]$ by semi-Lagrangian (SL) discrete approximations. Denote, $(\sqrt{a^\alpha_\delta})_i$ as columns of the square root of $a^\alpha_\delta=\mathcal{O}(\delta^{2-\sigma})$.

$$tr[a^\alpha_\delta D^2\phi]=\sum_{i=1}^N\frac{i_h\big[\phi(x+k(\sqrt{a^\alpha_\delta})_i)\big]+i_h\big[\phi(x-k(\sqrt{a^\alpha_\delta})_i)\big]-2\phi(x)}{k^2}+\mathcal{O}\Big(\frac{h^2}{k^2}\Big)+\mathcal{O}\big(\delta^{2(2-\sigma)}k^2\big).$$

We denote the local approximation by $\mathcal{L}^{\alpha,\delta}_{k,h}[\phi]$. To approximate the non-local term $\mathcal{I}^{\alpha,\delta}[\phi]$ we use the monotone interpolants i_h to approximate the integrands. We write the monotone approximation of $\mathcal{I}^{\alpha,\delta}$ as

$$\begin{aligned}\mathcal{I}^{\alpha,\delta}[\phi]&=\int_{|z|>\delta}i_h[\phi(x+z)-\phi(x)]\nu_\alpha(dz)+\mathcal{O}\left(\frac{h^2}{\delta^\sigma}\right)\\&=\sum_{\mathbf{j}\in\mathbb{Z}^N}\big(\phi(x+x_{\mathbf{j}})-\phi(x)\big)\kappa^{\alpha,\delta}_{h,\mathbf{j}}+\mathcal{O}\left(\frac{h^2}{\delta^\sigma}\right);\end{aligned}$$

where, $\kappa^{\alpha,\delta}_{h,\mathbf{j}}=\int_{|z|>\delta}\omega_{\mathbf{j}}(\eta^\alpha(z);h)\nu_\alpha(dz)$. Denote $\mathcal{I}^{\alpha,\delta}_h[\phi]$ as the nonlocal approximation term. The monotone numerical scheme is written as follows:

$$\sup_{\alpha\in\mathcal{A}}\Big\{f^\alpha(x)+c^\alpha(x)u_h(x)-\mathcal{L}^{\alpha,\delta}_{k,h}[u_h](x)-\mathcal{I}^{\alpha,\delta}_h[u_h](x)\Big\}=0.\quad(3)$$

Theorem(Rate of convergence)

Fix $\delta=h^{\frac{4}{4+\sigma}}$, let u be the viscosity solution of (1) and u_h the solution of the approximate equation (3).

(Strongly degenerate Case) Assume (**A**.1)-(**A**.3) hold. Then for any $0<\sigma<2$ we have

$$|u-u_h|\le C\,h^{\frac{4-\sigma}{4+\sigma}}.$$

(Weakly degenerate case) In addition, assume (**B**.2)-(**B**.3) hold. Then

$$|u-u_h|\le\begin{cases}C\,h^{\frac{4-\sigma}{4+\sigma}} & \text{for}\quad 0<\sigma\le1\\ C\,h^{\frac{\sigma(4-\sigma)}{4+\sigma}} & \text{for}\quad 1<\sigma<2.\end{cases}$$

Discussion:

- ✓ In strongly degenerate case, for $\sigma>1$ we proved sharper rates than the earlier existing results obtained for singular Lévy measures.
  - ✓ The rate decreases as the order of the nonlocal term increases for $\sigma>1$ and for σ near 2 the rate asymptotically approaches to $\mathcal{O}(h^{\frac13})$.
- ✓ For weakly degenerate case with better regularity structure on data and solution, we observe that the rate of convergence is always more than $\mathcal{O}(h^{\frac12})$, also the rate approaches to $\mathcal{O}(h^{\frac23})$ when $\sigma\to2$.

Approximation by power of discrete Laplacian

We consider this special discretization method by considering 'Fractional Laplacian' as the nonlocal term in equation (1). In particular, the equation takes the form:

$$\sup_{\alpha\in\mathcal{A}}\Big\{f^\alpha(x)+c^\alpha(x)u(x)+a^\alpha\,(-\Delta)^{\frac{\sigma}{2}}u(x)\Big\}=0,\quad(4)$$

The assumptions takes simpler form here due to this special structure. We consider discrete Laplacian as

$$\Delta_h[\phi](x)=\frac{1}{h^2}\sum_{i=1}^N\phi(x+e_ih)+\phi(x-e_ih)-2\phi(x).$$

Let $e^{t\Delta_h}\psi$ be the solution of semi-discrete heat equation

$$\begin{aligned}&\partial_tU(x,t)=\Delta_hU(x,t)\quad\text{for}\quad(x,t)\in\mathbb{R}^N\times(0,\infty)\\&U(x,0)=\psi(x)\quad\text{for}\quad x\in\mathbb{R}^N.\end{aligned}$$

Then the discretization of fractional Laplace is denoted by $(-\Delta_h)^{\frac{\sigma}{2}}$ and defined as

$$\begin{aligned}(-\Delta_h)^{\frac{\sigma}{2}}\phi(x)&:=\frac{1}{\Gamma(-\frac{\sigma}{2})}\int_0^\infty\Big(e^{t\Delta_h}\phi(x)-\phi(x)\Big)\frac{dt}{t^{1+\frac{\sigma}{2}}}\\&=\sum_{\mathbf{j}\in\mathbb{Z}^N\setminus\{0\}}\Big(\phi(x+x_{\mathbf{j}})-\phi(x)\Big)\kappa_{h,\mathbf{j}}.\end{aligned}$$

From the explicit representation of $e^{t\Delta_h}\phi$ we have that $\kappa_{h,\mathbf{j}}\ge0$, which ensures monotonicity of $(-\Delta_h)^{\frac{\sigma}{2}}\phi$. Furthermore, the *truncation error for the power of discrete Laplacian is of* $\mathcal{O}(h^2)$. Details can be found in [2, 1]. We now write the approximate equation as

$$\sup_{\alpha\in\mathcal{A}}\{f^\alpha(x)+c^\alpha(x)u_h(x)+a^\alpha(-\Delta_h)^{\frac{\sigma}{2}}[u_h](x)\}=0.\quad(5)$$

Remark 2. *For monotone difference quadrature approximation, the local truncation error is of order* $\mathcal{O}(\delta^{4-\sigma}+k^2\delta^{2(2-\sigma)}+\frac{h^2}{k^2}+h^2\delta^{-\sigma})\approx\mathcal{O}(h^{\frac{4-\sigma}{2}})$ *for* $\sigma>1$ *by the optimal choice of* k,δ. *Hence for* $\sigma>1$, *the maximum order of accuracy for such discretization is* $\frac32$. *Whereas, the novelty of choosing power of discrete Laplacian is that for any* $\sigma\in(0,2)$ *the consistency error is* $\mathcal{O}(h^2)$.

Theorem(Rate of convergence)

Let u be the viscosity solution of equation (4) and u_h the solution of the approximate equation (5).

(Strongly degenerate Case) Assume (**A**.2) and (**A**.1) hold. Then for any $\sigma\in(0,2)$ we have

$$|u-u_h|\le Ch^{\frac12}.$$

(Weakly degenerate Case) Assume (**B**.1) and (**B**.2) hold. Then

$$\|u-u_h\|_0\le\begin{cases}Ch^{\frac12} & \text{for}\quad 0<\sigma\le1\\ Ch^{\frac{\sigma}{2}} & \text{for}\quad 1<\sigma<2.\end{cases}$$

Discussion:

- ✓ For σ near 2, higher order accuracy than monotone difference quadrature scheme helps to get better rate of convergence in both strongly and weakly degenerate case.
- ✓ For $\sigma<1$, monotone difference quadrature scheme provides better convergence rate than power of discrete Laplacian.
- ✓ For weakly degenerate case The result is optimal for each σ near 2 under the given assumption. The convergence rate approaches to the rate obtained for 2nd order case (the rate is of $\mathcal{O}(h)$, c.f. [3]) when $\sigma\to2$.

Main Steps for weakly degenerate case

- The central idea of proving rate of convergence for both strongly and weakly degenerate equation is to use Krylov's regularizing argument ('shaking the coefficient method') [3].
- For weakly degenerate case, the major step is to show that

$$\|(-\Delta_h)^{\frac{\sigma}{2}}u_h\|_0\le K.$$

To see this we consider (5) with $c^\alpha=\lambda$ and using structure of the equation we can verify that $v=(-\Delta_h)^{\frac{\sigma}{2}}u_h$ satisfies

$$\lambda\,v+\sup_{\alpha\in\mathcal{A}}\Big\{(-\Delta_h)^{\frac{\sigma}{2}}[f^\alpha](x)+a^\alpha\,(-\Delta_h)^{\frac{\sigma}{2}}[v]\Big\}\ge0.\quad(6)$$

By regularity assumption (**B**.2) on f^α we get $\|(-\Delta_h)^{\frac{\sigma}{2}}[f^\alpha]\|_0\le F_2$. Therefore, $-F_2/\lambda$ would be a subsolution of (6). Then applying discrete comparison principle to (6) we get one sided bound

$$(-\Delta_h)^{\frac{\sigma}{2}}[u_h](x)\ge-\frac{F_2}{\lambda}.$$

For the upper bound, without loss of generality assume $(-\Delta_h)^{\frac{\sigma}{2}}[u_h](x)>0$ for each $x\in\mathbb{R}^N$, then by weak degeneracy condition (**B**.1)

$$\begin{aligned}\delta\,(-\Delta_h)^{\frac{\sigma}{2}}[u_h](x)\le\lambda\,u_h(x)&+\sup_{\alpha\in\mathcal{A}}\Big\{a^\alpha\,(-\Delta_h)^{\frac{\sigma}{2}}[u_h](x)+f^\alpha(x)\Big\}\\&+K\Big(\|u_h\|_0+\|f^\alpha\|_0\Big)\le K_1.\end{aligned}$$

- We define, $u_h^{(\epsilon)}=u_h*\rho_\epsilon$ where ρ_ϵ is a mollifier. To prove the precise rate for weakly degenerate case it is crucial to prove improved estimate on $\|u_h^{(\epsilon)}-u_h\|_0$.

By considering fractional heat kernel as the specific mollifier and using the consistency bound of $(-\Delta_h)^{\frac{\sigma}{2}}$ we prove

$$\|u_h^{(\epsilon)}-u_h\|_0\le K\Big(\epsilon^\sigma\|(-\Delta_h)^{\frac{\sigma}{2}}[u_h]\|_0+\omega(h)\Big).\quad(7)$$

Whereas, in general the estimate is $\|u_h^{(\epsilon)}-u_h\|_0\le K\epsilon\|u_h\|_{0,1}$.

- We regularize the equation (5) by taking 'fractional heat kernel' as mollifier. By noting $\|f^\alpha-(f^\alpha)^{(\epsilon)}\|_0\le K\epsilon^\sigma$ and using consistency error of $(-\Delta_h)^{\frac{\sigma}{2}}$ we get

$$\begin{aligned}&\lambda\,u_h^{(\epsilon)}+\sup_{\alpha\in\mathcal{A}}\Big\{f^\alpha(x)+a^\alpha(-\Delta)^{\frac{\sigma}{2}}u_h^{(\epsilon)}\Big\}\\&\le\epsilon^\sigma+Ch^2\Big(\|D^4u_h^{(\epsilon)}\|_0+\|u_h^{(\epsilon)}\|_0\Big).\end{aligned}$$

By using comparison principle for (4) and using the estimate (7) we have

$$u_h-u\le\frac{C}{\lambda}\Big(\epsilon^\sigma+h^2\|D^4u_h^{(\epsilon)}\|_0+h^2\|u_h^{(\epsilon)}\|_0\Big).$$

Other inequality follows by using similar arguments. Finally the result follows by using precise estimate of $\|D^4u_h^{(\epsilon)}\|_0$ and by optimal choice of ϵ.

References

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