Periodic gravity waves on water of finite depth

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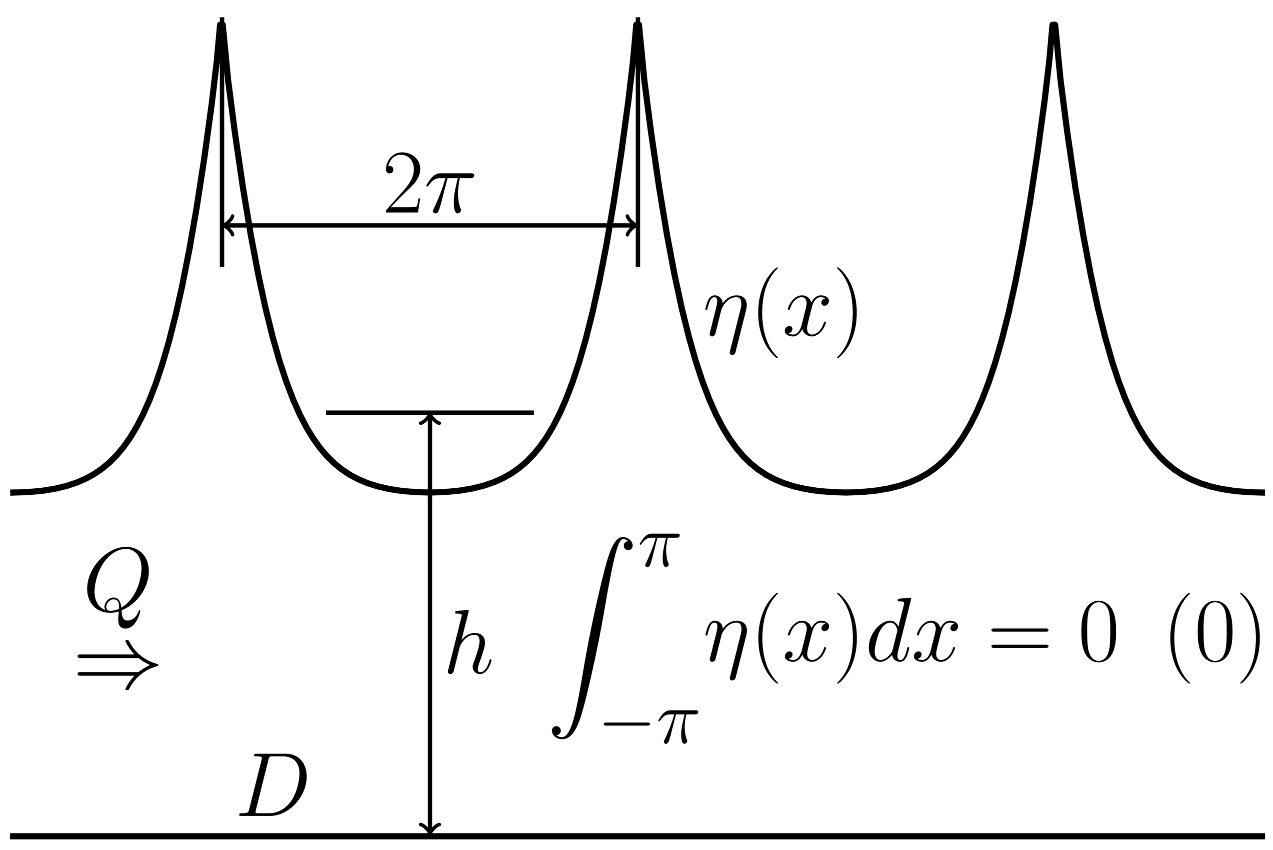
A non-dimensional formulation of the problem describing waves



For given flow rate Q and depth h we are looking for a triple (μ, η, ψ) satisfying

$$\psi_{xx} + \psi_{yy} = 0, \quad (x, y) \in D; \tag{1}$$
$$\psi(x, -h) = -Q, \quad x \in \mathbb{R}; \tag{2}$$
$$\psi(x, \eta(x)) = 0, \quad x \in \mathbb{R}; \tag{3}$$
$$|\nabla\psi(x, \eta(x))|^2 + 2\eta(x) = \mu, \quad x \in \mathbb{R}. \tag{4}$$

The squared Froude number $\mu = \pi c^2/(g\ell)$ gives the relation between its phase velocity c and period 2ℓ in the dimensional setting.

Problem (0)–(4) is equivalent to Babenko's equation with some $r \in (0,1)$, whose spectral form is

$$\mu J_r w = w + wJ_r w + \frac{1}{2}J_r(w^2), \tag{5}$$

where $\mu > 0$ is the same unknown as in (4) and $w \in W^{1,2}(0, \pi)$.

The self-adjoint operator $J_r = \sum_{n=1}^{\infty} \lambda_n P_n$ is defined for every conformal radius $r \in [0, 1)$, where P_n is the projector onto the subspace of $L^2(0, \pi)$ spanned by $\cos nt$ and $\lambda_n = n(1 + r^{2n})/(1 - r^{2n})$. The existence of small solutions of (5) follows from the Crandall–Rabinowitz theorem.

Modified Babenko's equation equivalent to equation (5)

Let us derive an equation, whose operators depend on the depth h instead of r and which is equivalent to (5).
First, we consider the nonlinear functional

$$r_h(w) = \exp\{-h - P_0 w\}. \tag{6}$$

Changing r to this functional in each λ_n, we obtain

$$\lambda_n^{(h)}(w) = n\frac{1 + [r_h(w)]^{2n}}{1 - [r_h(w)]^{2n}}, \quad n = 1, 2, \dots, \tag{7}$$

well defined provided $P_0 w \neq -h$. Then we put

$$\mathcal{J}_h w = \sum_{n=1}^{\infty} \left[\lambda_n^{(h)}(w)\right] P_n w, \quad w \in W^{1,2}(0, \pi), \quad P_0 w > -h.$$

In the same way, we define on $L^2(0, \pi)$ the nonlinear operator:

$$\mathcal{L}_h w = P_0 w + \sum_{n=1}^{\infty} \left[\mu_n^{(h)}(w)\right] P_n w, \quad \text{where} \quad \mu_n^{(h)}(w) = \frac{1 - [r_h(w)]^{2n}}{n\{1 + [r_h(w)]^{2n}\}}.$$

In terms of operators $\mathcal{J}_h$ and $\mathcal{L}_h$ defined for every $h > 0$, we write down the equation:

$$\mu(1 - P_0)w = \mathcal{L}_h w - \mathcal{L}_h(-w\mathcal{J}_h w) + \frac{1}{2}(1 - P_0)(w^2). \tag{8}$$

Proposition.
Let (μ, w), where $\mu > 0$ and $w \in W^{1,2}(0, \pi)$, be a solution of equation (5) with some fixed parameter $r \in (0, 1)$. Define $h = -\log r - P_0 w$. Then $h > 0$, w belongs to the domain of $\mathcal{J}_h$ and the pair (μ, w) satisfies (8).

On the contrary, let $h > 0$, and let $\mu > 0$ and $w \in W^{1,2}(0, \pi)$ with $P_0 w > -h$ solve (8). Then (μ, w) is a solution of (5) with $r = \exp\{-h - P_0 w\} \in (0, 1)$.

Tanaka's phenomenon

It is related to a turning point at the largest value of μ attained on the bifurcation curve C_1 for equation (5). This is related to the 'Tanaka instability' first found numerically and later investigated analytically.

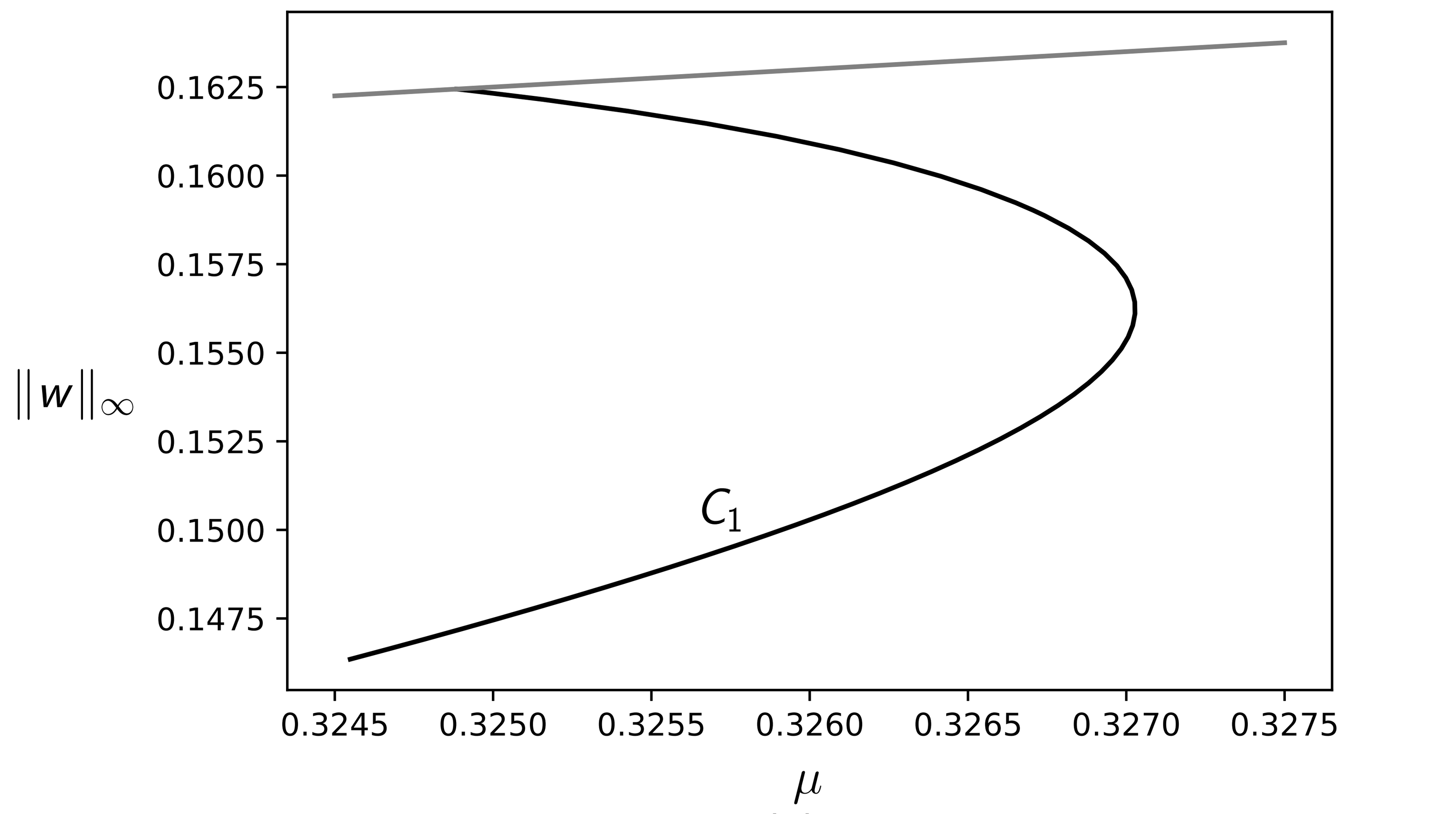


Figure : The solution branch C_1 for equation (5) with $r = 4/5$ in a vicinity of the turning point. The upper bound $\|w\|_\infty \leqslant \mu/2$.

An example of secondary bifurcation for equation (5)

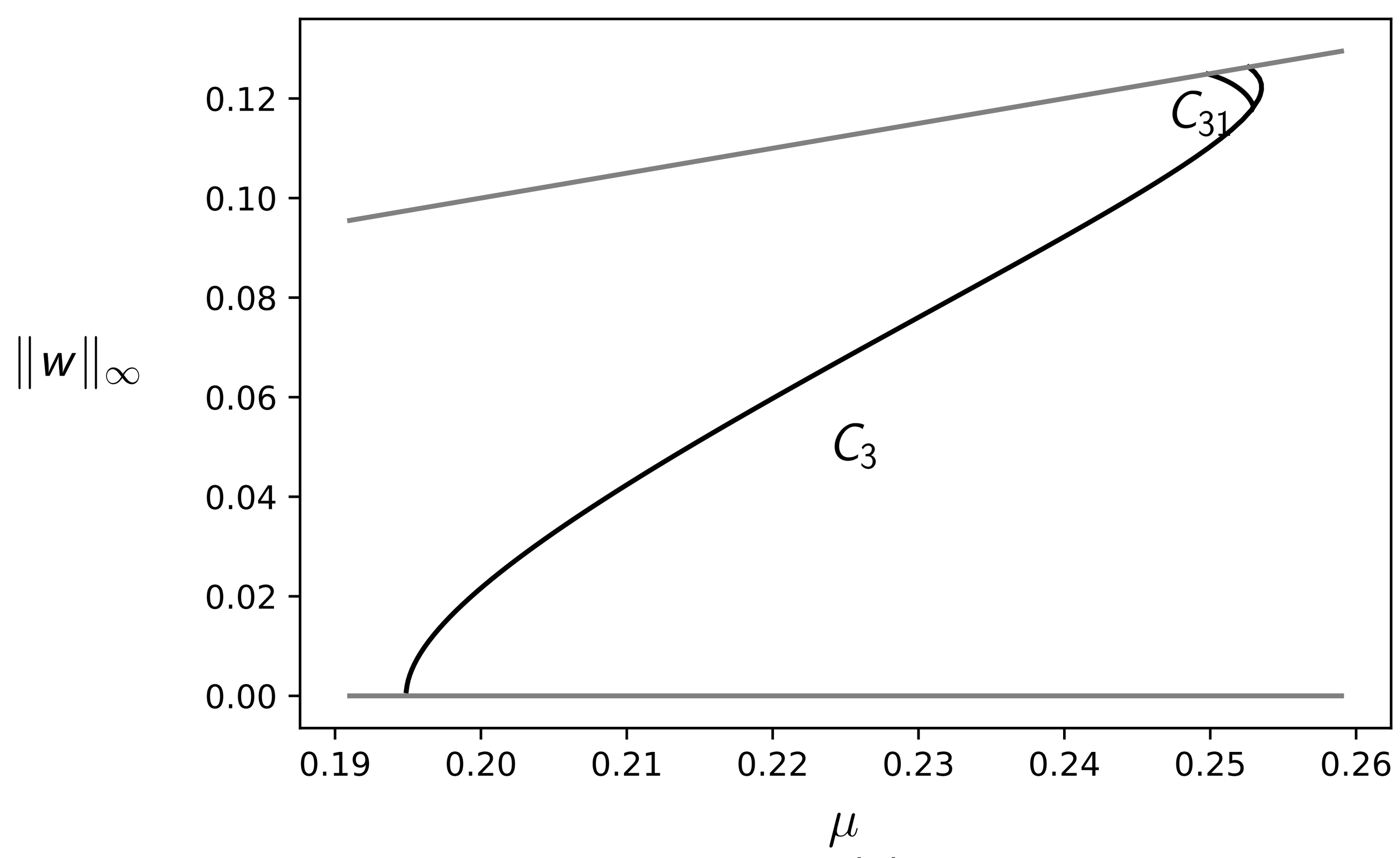


Figure : The branch C_3 of solutions of equation (5) with $r = 4/5$, bifurcating from the zero solution at $\mu_3(4/5) = 0.194868414381$. The secondary solution branch C_{31} bifurcates from C_3 at $\mu \approx 0.25298$. The upper bound $\|w\|_\infty \leqslant \mu/2$.

The wave profile corresponding to the end point on the branch C_{31}

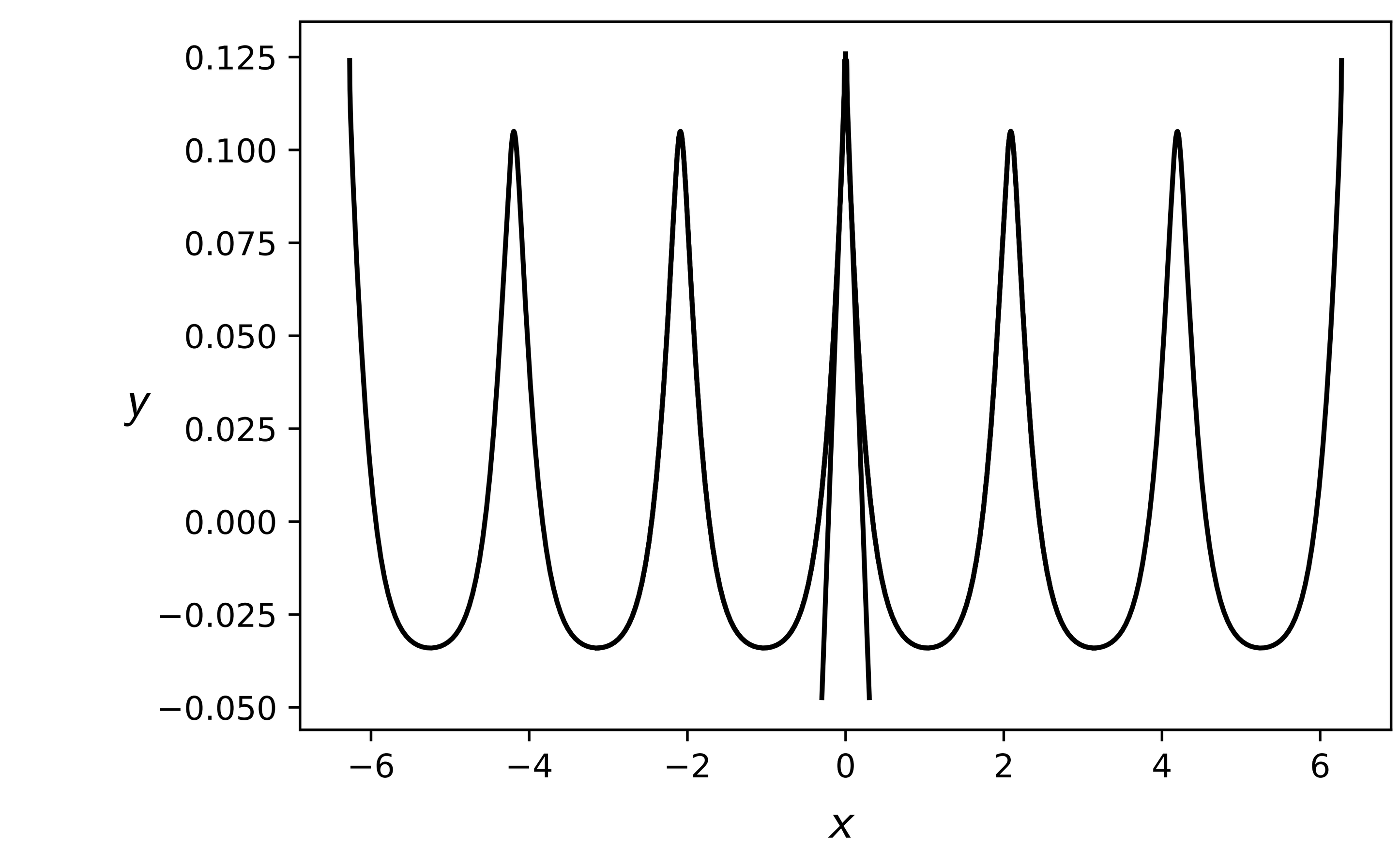


Figure : The wave profile of the extreme form corresponds to the end-point solution on the branch C_{31} for equation (5) with $r = 4/5$. Its characteristics are as follows: $\mu \approx 0.24827$ the profile's smooth crests (troughs) are at $y = \tilde{y}_c \approx 0.10406$ ($y = y_t \approx -0.03310$ respectively), whereas the peaks are at $y = \hat{y}_c \approx 0.12608$. The middle crest forms an angle equalled $2\pi/3$.